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TRANSIENT HEAT CONDUCTION IN INFINITE PLATES SITUATED IN A FOURTH-POWER RADIATIVE ENVIRONMENT

A THESIS

Presented to
the Faculty of the Graduate Division
by

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SUMMARY

In this work, the following physical problem is considered: An infinite plate of finite thickness, with constant properties, is initially at a uniform temperature. At a certain instant of time the plate begins to exchange energy at its surfaces by thermal radiation with a fixed environment. It is desired to determine the temperature at any point in the plate at any time.

Three methods of approach are employed in connection with the above problem: (a) an approximate solution based on the heat balance integral technique, (b) a numerical solution which was programmed for use on a digital computer, and (c) an experimental solution.

The numerical solution is estimated to be accurate to ± 0.1 per cent within the range of parameters considered. The accuracy of the approximate solution is estimated at ± 10 per cent, except for a peculiarity existing for a short period near the center of the plate, for which the error may be as great as ± 50 per cent. The experimental solution, which was performed for three sets of values of the parameters involved, gave accuracy within ± 2 per cent, as compared with the numerical solution.

In the experimental solution, the temperature-time histories of both surfaces of the plate, and of its center,

were measured. In order to compare these with the numerical and approximate solutions, certain combinations of properties of the plate and its environment were measured. This was done by special techniques using the same plate and the same experimental apparatus from which the temperature-time histories were obtained.

Several secondary effects were considered, including the effect of conduction and convection in an ambient medium surrounding the plate.

LIST OF SYMBOLS

A	Surface area, ft.^{2}
α	Thermal diffusivity, ft.^{2}/hr.
c	Specific heat, Btu/lb.$_m$ ^oR
C	Conductive or convective coefficient, Btu/hr.ft.^{2} ^oR
δ	Location of point where temperature begins to change, Fig. 3
ϵ	Emissivity
$\sigma\mathcal{F}$	Gray body shape factor between body and environment
G	Dimensionless parameter, equation (16)
h	Space interval
H	Dimensionless parameter, equation (22)
I	Heater current, amp.
J	Dimensionless parameter, equation (128)
k	Time increment
k	Thermal conductivity, Btu/hr. ft. ^oR
K	Dimensionless parameter, equation (129)
L	Half thickness of plate, ft.
p	Pressure, lb.$_f$/ft.^{2} abs.
p_μ	Pressure, μ Hg. abs.
P_w	Heater power, watt
q	Heat flux, Btu/hr. ft.^{2}
q_o	Constant heat input to surface, Btu/hr. ft.^{2}
q_s	Heat flux at surface, Btu/hr. ft.^{2}

LIST OF SYMBOLS (Continued)

Q	Total heat transferred, Btu
$\dot{Q}$	Total heat transfer rate, Btu/hr.
R	Heater resistance, ohm
R_g	Gas constant, ft. $lb._f/lb._m$ ^oR
ρ	Density, $lb._m/ft.^3$
σ	Stefan-Boltzmann constant = 0.1714×10^{-8} Btu/hr. ft.^{2} $^oR^4$
t	Dimensionless temperature
t_e	Dimensionless environmental temperature
T	Temperature, ^oR
T_s	Surface temperature, ^oR
T_o	Initial temperature, ^oR
T_e	Environmental temperature, ^oR
T_h	Temperature of hot face in measuring $\sigma\mathcal{F}L/k$, ^oR
T_c	Temperature of cold face in measuring $\sigma\mathcal{F}L/k$, ^oR
τ	Dimensionless time, equation (14)
$\Delta\tau$	Time increment
θ	Time, hr.
v	Heater voltage, volt
V	Volume, ft.^{3}
x	Distance coordinate, ft.
ξ	Dimensionless distance, equation (15)
$\Delta\xi$	Space increment
z	Dimensionless surface temperature

LIST OF SYMBOLS (Continued)

z_e Dimensionless environmental temperature

Symbols not included in the above list are defined in the text. The subscript "w" refers to the heater wire. The subscript "s" refers to the surface of the sample. In most cases, the symbol without subscript refers to the sample.

CHAPTER I

INTRODUCTION

The Generic Problem.--In the operation of space satellites and other space vehicles, the only available means for transferring large quantities of energy to or from the vehicle is by thermal radiation. In certain cases it may be desirable to dissipate energy from the spacecraft by radiation to outer space. In other cases, the radiation incident upon the vehicle, from the sun or other source, must be allowed for in its design. In all events, the existence of thermal radiation must be considered in the design, operation, and maintenance of space vehicles.

In many applications the transient behavior of the space vehicle, or of certain of its components, under the effect of incident or emitted thermal radiation is of primary importance. For example, when an earth satellite enters the earth's umbra, the mean radiant temperature of the satellite's environment is suddenly reduced, giving rise to a transient cooling of the satellite. This process is reversed when the satellite leaves the earth's umbra. Since the total heat capacity of artificial satellites is normally small, it is expected that large periodic temperature changes thus occur.

In connection with the study of such transient phenomena, the following generic problem becomes of interest. Consider a body which can transfer heat internally by conduction only, and whose external surfaces are exposed to an environment with which energy may be exchanged by means of thermal radiation. In addition, if the body is hollow, heat may be transferred by radiation and convection to and from the interior surfaces of the body. Initially the body has a uniform temperature throughout. At a certain instant of time, energy interchange by thermal radiation with the environment commences, and continues until the temperature of the body is again uniform, this time at the mean radiant temperature of the environment. It is desired to determine the temperature at any point in the body at any time.

There exist other applications of the above problem, a few of which will now be mentioned. In steering space vehicles, a likely method is by the use of small rockets which are fired intermittently (1). Since the temperatures of rocket exhausts are much higher than existing materials can withstand, firing must cease whenever the temperature of the exhaust nozzle reaches the limit which its material can safely withstand. To prepare for the next firing, the exhaust nozzle may be cooled by thermal radiation into space. This cooling process represents a specific case of the problem stated in the preceding paragraph.

The following extension, which is mathematically equivalent, leads to the following important practical application. In addition to the energy exchange by thermal radiation between the external surfaces of the body and its environment, which occurs in the generic problem, energy is added to the external surfaces from an independent source. This suggests the important practical problem of transient aerodynamic heating of a body which can lose energy by radiation.

Examples are by no means limited to space vehicles. An example is that of a body being heated in a furnace. Under certain conditions, the bulk of the energy interchange between the furnace and the body will consist of thermal radiation between the furnace walls and the body surfaces. Also, in cooling bodies initially at very high temperatures, the bulk of the heat loss may well be due to thermal radiation, at least until the body surface is cooled down to moderate temperatures. In vacuum processing of materials, where radiation is the only significant mode of heat transfer, problems involving heating and cooling of bodies are necessarily special cases of the problem considered in this section.

To solve the generic problem in complete generality imposes severe if not impossible mathematical difficulties. Consequently, special cases which involve simplifying assumptions are initially considered. The first such assumption is that the properties of the body--its thermal conductivity, thermal diffusivity, and surface emissivity--

are constant within the body and on its surface, and do not change either with time or with temperature. The second assumption is that each point on the body surface "sees" the same environmental mean radiant temperature, and that the shape factor connecting the point with its environment is the same for all points on the body surface. The third assumption relates to the body shape. The simplest such assumption is that of a semi-infinite body. Other body shapes which appear to be tractable in terms of available mathematical methods are, in apparent order of difficulty: the infinite plate of finite thickness; the infinite circular cylinder; the sphere; the hollow infinite circular cylinder, with a variety of boundary conditions on its interior surface. Other shapes could be added to this list; however, their mathematical solution is thought to be extremely difficult, and consequently attention should currently be confined to the more elementary shapes listed above.

After presenting a review of the literature concerned with various aspects of the generic problem, the special case of the infinite plate of finite thickness will be solved, under the first and second assumptions listed in the preceding paragraph.

Review of Literature.--A fair amount of work has already been done toward solving various special cases of the problem discussed in the preceding section. There are a number of references extant in the literature concerned with various special cases of the problem. These will be considered in detail in this section.

Jaeger (2) considered the case of a semi-infinite body, initially at a uniform temperature, which is suddenly exposed to a heat flux on its free surface according to the law:

$$q_s = H\ T_s^{\ m}\ , \tag{1}$$

in which H and m are constants. An exact solution in series for the temperature as a function of position and time was obtained. However, this series converges so slowly at large time values that the solution is useful only for small values of time. This is one of the very few exact solutions available to a special case of the generic problem; most of the work done has been devoted to obtaining approximate solutions.

Several approximate techniques have proven fruitful or at least give promise of doing so. These techniques will now be considered in turn.

Probably the simplest approximate technique to apply is the heat balance integral technique developed by Goodman (3, 4, 5) and by Reynolds and Dolton (6). This method also has the advantage that a general formula for temperature as a function of position and time is obtained. The method is somewhat analogous to the Karman integral method so useful in boundary layer theory. The heat balance integral technique is applicable to a wide variety of transient conduction problems. Goodman (3) applied the technique to the problem of the semi-infinite body discussed previously, and compared the results obtained with the exact solution derived by Jaeger (2). Chambre (7) applied the integral technique to the same problem, in order to obtain a starting solution for an iterative scheme which will be considered later. Schneider (8) considered the technique in connection with both the semi-infinite body and the infinite plate of finite thickness. However, he made no attempt to evaluate certain integrals appearing in his solution, except for a few specific numerical values of the parameters involved, and thus he did not obtain a general solution to either problem.

An approximate solution was obtained by Robbins (1), by assuming an infinite thermal conductivity for the body material. His interest was in the cooling of rocket nozzles by thermal radiation into space during periods when the rocket is not being fired. The paper by Schneider (8), alluded to previously, was written primarily to estimate the error inherent in this assumption of infinite thermal conductivity.

The use of the Laplace transform method to problems of the type under consideration leads to a nonlinear Volterra integral equation for the temperature at the surface. By means of Duhamel's theorem, the temperature at any point in the body at any time may be obtained if the surface temperature history is known. Some consideration has been given in the literature to the problem of solving the nonlinear Volterra integral equation for the surface temperature as a function of time.

Abarbanel (9, 10) has considered this problem for several body shapes, with noteworthy success. He obtained the integral equation satisfied by the surface temperature for the semi-infinite solid, the infinite plate of finite thickness, the solid sphere, and the hollow sphere or spherical shell. In each of these cases he first considered an environmental temperature of 0 ^{o}R, and then considered an arbitrary, constant environmental temperature. For each of these cases he obtained short-time and long-time asymptotic solutions to the surface temperature. In addition, he also found a general analytical solution to the surface temperature for the problem of a semi-infinite plate radiating to an environment at 0 ^{o}R, by means of an iterative technique. However, like Jaeger's solution (2), this solution converges very slowly except for small time values. In all the cases considered, Duhamel's theorem was applied to obtain the temperature at any point in the body in terms of the surface temperature.

Chambre (7) considered the case of a semi-infinite body, initially at a uniform temperature, which suddenly begins to exchange heat with its environment in such a way that the following rather general nonlinear boundary condition holds:

$$q_s = \beta \, f(T_s) \, , \tag{2}$$

where β is a constant and f is an arbitrary analytic function of its argument. The use of Laplace transforms leads to the following singular, nonlinear, Volterra integral equation:

$$T_s(\theta) = T_o - \frac{\beta}{\sqrt{\pi}} \int_o^{\theta} f\left[T_o(\eta)\right] \frac{d\eta}{\sqrt{\theta-\eta}} \, . \tag{3}$$

The author then discussed the iterative solution to equation (3). As mentioned previously, this same author used the results of the heat balance integral technique to obtain a starting solution for this iterative process.

Stickler (11) considered the problem of an infinite plate of finite thickness which radiates to an environment of fixed temperature, and which also receives energy at its surface at a constant rate from an independent source. He derived the integral equation for the surface temperature, and set up an iterative scheme for its solution. He was able to perform the first iteration explicitly, and found some error bounds for the result of the first iteration.

Questions of existence, uniqueness, and convergence of the solution of an integral equation similar to equation (3) were discussed in papers by Mann and Wolf (12), Roberts and Mann (13), and Padmavally (14). Friedman (15) gave proofs of existence, uniqueness, and boundedness of the solution of transient heat conduction problems under general initial and boundary conditions, which include radiation boundary conditions as special cases.

Several well-known numerical techniques may be applied to special cases of the generic problem. All of these methods are based upon the approximation of the derivatives in the differential equation and boundary conditions by finite-difference ratios. All of these methods share the disadvantage that, unless it is possible to remove all the parameters in the problem by reducing to nondimensional form, the numerical procedure must be repeated in its entirety for every new set of parameters.

Probably the best-known numerical technique in connection with transient heat conduction problems is the Schmidt method. The application of this method to the semi-infinite body with radiation boundary condition was discussed in detail by Jaeger (2). An instrument for the graphical application of Schmidt's method is described by Jaeger (16). However, this instrument has no direct provision made for treating the radiation boundary condition.

A more elaborate numerical method has been developed by Crank and Nicolson (17), using more accurate finite-difference approximations. Their method, which is an implicit numerical method, involves larger time steps than the Schmidt method for the same accuracy, but, on the other hand, each time step is more laborious to perform. Thus, the time and labor required to solve a given problem by Crank and Nicolson's method is probably not vastly different from that needed to solve the same problem by Schmidt's method.

Eyres, *et al*. (18) set up finite-difference approximations to transient heat conduction problems, including the radiation boundary condition. Their particular interest was in setting up the problem for use on the differential analyzer. Crank (19) devoted a portion of a chapter to consideration of some numerical methods, mainly Schmidt's method and Crank and Nicolson's method.

Lowan (20) obtained some error bounds for the propagation of round-off errors in numerical solutions of transient heat conduction problems, where a certain finite-difference equation is used to approximate the heat conduction equation. Freed and Rallis (21) devised a scheme for estimating the error inherent in replacing the differential system by a particular type of difference system. Such error investigations may prove useful in deciding whether or not a particular numerical solution is practical, particularly for use on a computer.

The calculus of variations offers a possible approach to the solution of special cases of the generic problem. The variational principle for heat conduction was developed by Chambers (22). Its application to practical heat conduction problems was discussed in detail by Biot (23). Although he did not solve any problems involving the radiation boundary condition, Biot did however point out a modification of his procedure which would lead to an approximate solution to such a problem.

Analogy methods form another approach which shows considerable promise for solving the generic problem. Several electrical analogies appear to be adaptable to the nonlinear radiation boundary condition. Lawson and McGuire (24) have described a resistance-capacitance network. In their system, the radiation boundary condition could be simulated by use of a substance called "metrosil," which has a resistance proportional to a constant power of the applied voltage.

Liebmann has invented a promising electrical resistance analog, as described in references (25, 26, 27). The analogy actually produces a solution to the finite-difference approximation to the transient heat conduction problem. Basically, the network is such that if the voltage analogous to the temperature distribution at time $n\Delta\theta$ is fed into one side of the network, the resultant <u>steady-</u>

state output on the other side is proportional to the temperature distribution at time (n+1) $\Delta\theta$. This obviates the difficulties inherent in a transient electrical analogy, and also allows for changing the parameters of the system and for changing the boundary conditions as the solution proceeds. However, the system does require the step-by-step transfer of output to input to obtain a solution. Provision can be made in the system for treating complicated boundary conditions, including the radiation boundary condition, as well as variable parameters such as thermal conductivity, and also to allow for heat generation in the medium. The system can also be adapted to treat two- and three-dimensional problems, various body configurations, as well as other effects. This device seems very promising with respect to the generic problem.

Zerkle (28) has solved the problem of an infinite plate of finite thickness with the radiation boundary condition on a passive electric analog computer. He simulated the radiation boundary condition using a function generator, by replacing the nonlinear surface temperature with a series of straight line segments.

A hydraulic analogy also seems promising. Moore (29) described such an analogy, using liquid flow to represent heat flow and pressure to represent temperature. Coyle (30, 31) used the same analogous quantities, except that he used air (or some other gas) instead of liquid. Although neither author provided for a nonlinear boundary condition, the radiation boundary condition could likely be simulated by a special nonlinear valving arrangement. Like the electrical analogy, this analogy holds considerable promise for solving the generic problem.

There are some additional references available which do not bear directly on the problems of interest here, but which may nonetheless prove useful in their analysis. The problem of a body, initially at a uniform temperature, which is suddenly allowed to exchange heat at its surface according to the law:

$$q_s = f(\theta) , \tag{4}$$

is one that should be considered. Sutton (32) considered this problem for an infinite plate of finite thickness, insulated on the opposite face. Poritsky and Powell (33) considered this problem for a semi-infinite body with heat transfer on the free surface given by:

$$q_s = \theta^n/n!. \tag{5}$$

Carslaw and Jaeger (34) considered a number of problems of this sort.

Another problem of interest is that of the heating or cooling of a diathermic solid by radiative transfer. In this problem, absorption and emission of radiant energy occur throughout the solid, not merely on its surface. This problem has been tackled by Hottel and Williams (35), Gardon (36), and Gardon and Michalik (37). Van der Held (38) has considered this diathermancy effect with respect to solids which are usually considered opaque.

<u>Mathematical Formulation</u>.--The purpose of this work is to solve the following special case of the generic problem stated previously (see Fig. 1). An infinite plate of thickness 2L is initially at a uniform temperature T_o. At time zero, energy interchange by thermal radiation commences to an environment at an absolute temperature T_e. The environmental temperature and shape factor are assumed to be constant and to be the same for both faces of the plate. It is desired to determine the temperature in the plate as a function of position and time, for all points in the plate and for all time greater than zero.

The physical problem stated in the preceding paragraph will be represented here mathematically by the following initial- and boundary-value problem. It is assumed that the properties of the plate material--thermal conductivity and thermal diffusivity--are constant. The differential equation to be satisfied is:

$$\frac{1}{\alpha}\frac{\partial T}{\partial \theta} = \frac{\partial^2 T}{\partial x^2}\,, \qquad (6)$$

$$T = T(x,\theta),\ -L < x < L,\quad \theta > 0\ .$$

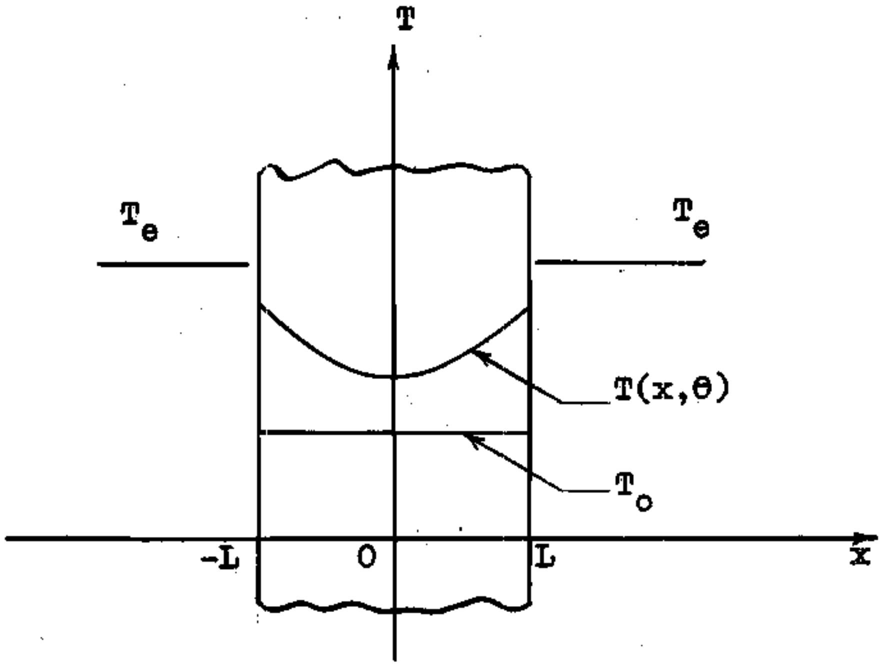


Figure 1

Section of the Infinite Plate

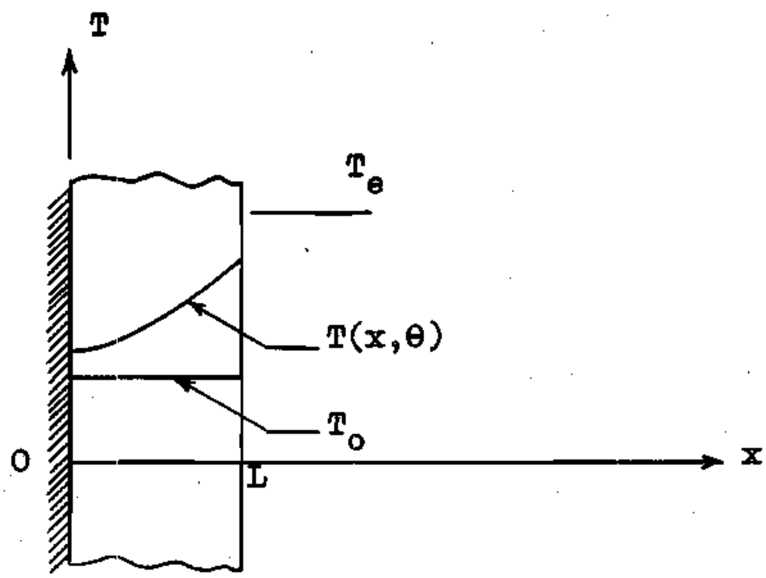


Figure 2

Section of the Infinite Plate, with Insulated Boundary

The initial condition is:

$$T(x,0) = T_o \ , \quad -L \leq x \leq L \ , \tag{7}$$

whereas the boundary conditions are:

$$\frac{\partial T}{\partial x}(-L,\theta) = -\frac{1}{k}\sigma \mathcal{F}\left[T_e{}^4 - T^4(-L,\theta)\right] \ , \tag{8}$$

$$\theta > 0 \ ,$$

$$\frac{\partial T}{\partial x}(L,\theta) = \frac{1}{k}\sigma \mathcal{F}\left[T_e{}^4 - T^4(L,\theta)\right] \ , \tag{9}$$

$$\theta > 0 \ .$$

Inasmuch as conditions are the same on the two faces of the plate, the plate is heated symmetrically; hence the above physical problem may be replaced by the following physical problem (see Fig. 2). An infinite plate of thickness L is initially at a uniform absolute temperature T_o. At time zero, energy interchange by thermal radiation commences between one face of the plate and an environment at an absolute temperature T_e, while the opposite face of the plate remains insulated. The environmental temperature and shape factor are assumed to remain constant, as are the thermal conductivity and thermal diffusivity of the plate material. It is then desired to determine the temperature in the plate as a function of position and time.

The mathematical problem expressed in equations (6)-(9) is now altered to fit the equivalent physical problem. The differential equation becomes:

$$\frac{1}{\alpha}\frac{\partial T}{\partial \theta} = \frac{\partial^2 T}{\partial x^2}\ , \tag{10}$$

$$T = T(x,\theta)\ ,\ 0 < x < L,\ \ \theta > 0\ ;$$

the initial condition is:

$$T(x,0) = T_o,\ \ 0 \leq x \leq L\ ; \tag{11}$$

and the boundary conditions are now:

$$\frac{\partial T}{\partial x}(0,\theta) = 0\ ,\ \ \theta > 0, \tag{12}$$

$$\frac{\partial T}{\partial x}(L,\theta) = \frac{1}{k}\sigma\, {}^{\sigma}\!\mathcal{F}\left[T_e{}^4 - T^4(L,\theta)\right]\ , \tag{13}$$

$$\theta > 0\ .$$

In equations (10)-(13), α, k, ${}^{\sigma}\!\mathcal{F}$, L, T_o, and T_e are constant.

The solution of the system of equations (10)-(13) is the primary purpose of this work. In anticipation of the use of numerical methods in the solution, and also to better present the results of the solution once it is obtained, as many as possible of the parameters α, k, ${}^{\sigma}\!\mathcal{F}$, L, T_o, and T_e, will be eliminated by reducing the system of equations to nondimensional form. For this purpose, define:

$$\gamma = \alpha \theta / L^2 , \qquad (14)$$

$$\xi = x/L , \qquad (15)$$

$$t = T/T_o . \qquad (16)$$

With these substitutions, the system of equations (10)-(13) becomes:

$$\frac{\partial t}{\partial \gamma} = \frac{\partial^2 t}{\partial \xi^2} , \qquad (17)$$

$$t = t(\xi, \gamma), \quad 0 < \xi < 1, \quad \gamma > 0,$$

$$t(\xi, 0) = 1, \quad 0 \leq \xi \leq 1 , \qquad (18)$$

$$\frac{\partial t}{\partial \xi}(0, \gamma) = 0, \quad \gamma \geq 0 , \qquad (19)$$

$$\frac{\partial t}{\partial \xi}(1, \gamma) = G - Ht^4(1, \gamma), \quad \gamma > 0 . \qquad (20)$$

In equation (20) the following dimensionless parameters have been defined:

$$G = \frac{\sigma \mathcal{F} L T_e^{\,4}}{k T_o} , \qquad (21)$$

$$H = \frac{\sigma \mathcal{F} L T_o^{\,3}}{k} . \qquad (22)$$

If, in addition to the radiant interchange with the environment occurring at the surface $x = L$ in Fig. 2, a constant heat input q_o per unit time and area, from an independent source is included, the expression for G in equation (20) becomes:

$$G = \frac{\sigma \mathcal{F} L T_e^{\,4}}{k T_o} + \frac{q_o L}{k T_o} \,. \tag{23}$$

The expression for H remains the same.

Solutions to the system of equations (17)-(20) are obtained in succeeding chapters. The range of values of the parameters G and H for which a solution is desired is $0 < G, H < \infty$. In practice, a much narrower range than this will suffice.

<u>Plan of Attack</u>.--In solving the system of equations (17)-(20), three approaches are presented. The first approach, which is covered in Chapter II, is that of the heat balance integral technique. This method was discussed in references (3-8) and applied to the semi-infinite body with radiation boundary condition in references (3, 7, 8). Schneider (8) also applied the method to the infinite plate, but did not carry the method through to a general solution. Instead, he was content to leave the solution in terms of certain integrals which he anticipated would be evaluated numerically. It is the purpose of Chapter II to derive an approximate general solution to the infinite plate problem by the heat

balance integral technique, and to present it in a form useful for computational purposes.

The second approach, which is developed in Chapter III, is a numerical solution based on finite-difference approximations to the derivatives appearing in the differential equation and boundary conditions. Although the numerical methods of references (2, 17, 18) are applicable with suitable modification, the actual method chosen differs from each of these. The method developed in Chapter III was used on a digital computer, and it was necessary to devise a method which gives a reasonably accurate answer in a reasonable amount of computer time.

The third approach to the infinite plate problem is an experimental solution wherein the infinite plate is replaced by a finite plate whose thickness is small compared to its lateral dimensions. The experimental apparatus and procedures are discussed in Chapter IV, in which also are derived the equations for the necessary corrections and measurement of properties required in performing the experiment. Also, the criteria,under which certain extraneous effects are negligible, are included in Chapter IV.

In Chapter V are presented the numerical results of the three methods of solution discussed above. In addition, the numerical results from Zerkle (28) along with the asymptotic solutions of Abarbanel (10), are presented for a few sets of values of the dimensionless parameters G and

H. Also, the effect of varying the pressure of the medium surrounding the plate is discussed. A discussion is given in this chapter of certain errors inherent in the experimental solution.

CHAPTER II

APPROXIMATE SOLUTION

The Heat Balance Integral Technique.--The heat balance integral technique affords an approximate solution to any of a large class of transient heat conduction problems. This method was applied in references (3, 4, 6, 7, 8) to various problems. The method is analogous to the Karman integral method for boundary layer flow.

In the heat balance integral technique, the temperature distribution within the body in question is approximated by a simple expression. If a polynomial expression is used, it will have the form:

$$t(\xi, \gamma) = A_o(\gamma) + A_1(\gamma)\xi + A_2(\gamma)\xi^2 + \dots \quad . \tag{24}$$

The coefficients A_o, A_1, A_2, . . . in equation (24) are determined in the following manner. First, equation (24) must satisfy the boundary conditions appropriate to the problem. Second, the total energy content of the body as determined from equation (24) must equal the total energy input into the body by radiation. This condition, as will be seen, leads to an integral equation, which accounts for the name "heat balance integral technique." Third, if additional conditions are needed to evaluate all the coefficients in equation (24), these conditions may be obtained by combining the boundary conditions of the given problem with equation (17), written at the corresponding boundary.

Once the coefficients in equation (24) have been evaluated, the problem is essentially solved. From the approximate temperature distribution, approximations to other quantities of interest may be readily obtained.

In applying the heat balance integral technique to the problem expressed in equations (17)-(20), the solution will be separated into two parts: the starting solution and the continued solution. The starting solution holds for small time values, during which the plate behaves essentially as a semi-infinite body having the same initial temperature distribution. The continued solution holds for larger time values, after the effect of the finite plate thickness becomes evident.

<u>The Starting Solution</u>.--As mentioned above, the starting solution holds during small time values, when the infinite plate of finite thickness behaves essentially as a semi-infinite body. This period ends when the temperature at the insulated face of the plate begins to change appreciably. During this period, in applying the heat balance integral technique, it is convenient to treat the finite plate as a semi-infinite body. Thus, during this period the mathematical problem expressed by equations (17)-(20) is replaced by the following:

$$\frac{\partial t}{\partial \tau} = \frac{\partial^2 t}{\partial \xi^2}, \tag{25}$$

$$t = t(\xi, \tau), \quad \xi < 1, \quad \tau > 0;$$

$$t(\xi, 0) = 1, \quad \xi < 1; \tag{26}$$

$$\frac{\partial t}{\partial \xi}(1, \tau) = G - Ht^4(1, \tau), \quad \tau > 0. \tag{27}$$

In addition, the temperature $t(\xi, \tau)$ is bounded as $\xi \rightarrow -\infty$.

Goodman (3), Chambre (7), and Schneider (8) have each applied the heat balance integral technique to the problem of equations (25)-(27). Although each of these authors indicated the procedure necessary to obtain a solution, neither of them actually carried the solution through to its completion. Also, the coordinate systems used in their papers differ from that used in equations (25)-(27). For these reasons, and also because of its importance in the development of the continued solution, the starting solution will be derived in detail in this section.

Define for convenience:

$$z(\tau) = t(1, \tau), \tag{28}$$

so that equation (27) becomes:

$$\frac{\partial t}{\partial \xi}(1,\gamma) = G - Hz^4 . \qquad (29)$$

At any time γ, there exists a point in the semi-infinite body where the temperature is on the verge of changing by a significant amount. The depth of this point from the surface is known as the penetration depth. The temperatures of all points farther from the surface will have changed by only a negligible amount up to this time. Put another way, the amount of energy which has been conducted to depths greater than the penetration depth is negligible. This penetration depth, which is a function of time γ, is given by $(1 - \delta(\gamma))$ in Fig. 3.

It is apparent from Fig. 3 that:

$$t\left[\delta(\gamma),\gamma\right] = 1 . \qquad (30)$$

In a homogeneous solid, both $t(\xi,\gamma)$ and $\partial t(\xi,\gamma)/\partial\xi$ are continuous functions of ξ and γ. Hence:

$$\frac{\partial t}{\partial \xi}\left[\delta(\gamma),\gamma\right] = 0 . \qquad (31)$$

Differentiating equation (30) with respect to time gives:

$$\frac{dt}{d\gamma}\left[\delta(\gamma),\gamma\right] = \frac{\partial t}{\partial \xi}\left[\delta(\gamma),\gamma\right]\frac{d\delta(\gamma)}{d\gamma} + \frac{\partial t}{\partial \gamma}\left[\delta(\gamma),\gamma\right] = 0 .$$

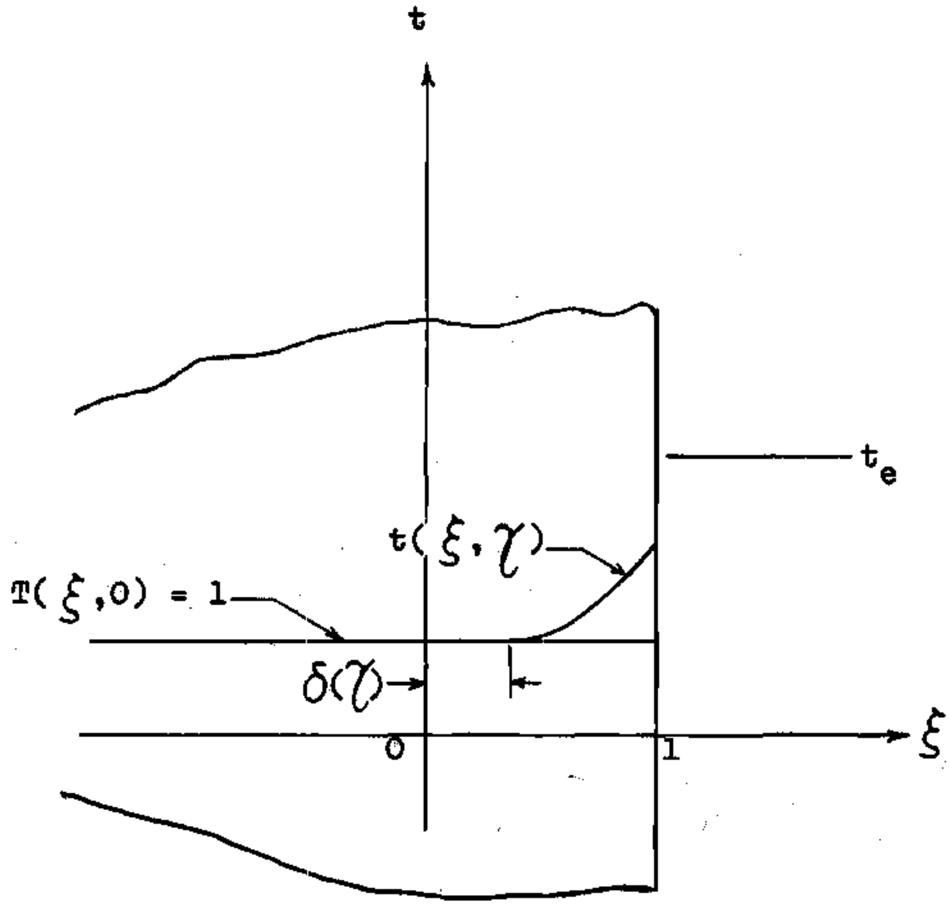


Figure 3

Section of the Semi-Infinite Body

Using equation (31), this results in:

$$\frac{\partial t}{\partial \gamma}\left[\delta(\gamma),\gamma\right] = 0 . \tag{32}$$

Then evaluating equation (25) at $\xi = \delta(\gamma)$ and applying equation (32) gives:

$$\frac{\partial^2 t}{\partial \xi^2}\left[\delta(\gamma),\gamma\right] = 0 . \tag{33}$$

Equations (28)-(31) and (33) are the boundary conditions which will be considered in evaluating the coefficients in the approximate polynomial expression for $t(\xi,\gamma)$.

The following polynomial approximation for $t(\xi,\gamma)$ is given:

$$t(\xi,\gamma) = 1 + \frac{(z-1)\left[\xi - \delta(\gamma)\right]^3}{\left[1-\delta(\gamma)\right]^3} ; \tag{34}$$

it may be easily verified that equations (28), (30), and (33) are satisfied by this polynomial. Substituting equation (34) into equation (29) leads to the result:

$$\delta(\gamma) = 1 - \frac{3(Z-1)}{G - Hz^4} . \tag{35}$$

Next, an integral equation in the temperature $t(\xi,\gamma)$ is obtained. Integrating equation (25) with respect to ξ gives:

$$\int_1^{\delta(\tau)} \frac{\partial t}{\partial \tau}(\xi,\tau)d\xi = \frac{\partial t}{\partial \xi}\left[\delta(\tau),\tau\right] - \frac{\partial t}{\partial \xi}(1,\tau).$$

Interchanging the order of differentiation and integration, and using equations (30) and (31) gives:

$$\frac{d}{d\tau}\int_{\delta(\tau)}^{1} t(\xi,\tau)d\xi + \delta'(\tau) = \frac{\partial t}{\partial \xi}(1,\tau) . \qquad (36)$$

Using equation (29), equation (36) reduces to:

$$\frac{d}{d\tau}\int_{\delta(\tau)}^{1} t(\xi,\tau)d\xi = -\delta'(\tau) + G - Hz^4 . \qquad (37)$$

In question (37), which is called the "heat balance integral," the integral is proportional to the total energy content at time τ.

Substituting equation (34) into equation (37), performing the indicated integration and differentiation, leads to the equation:

$$\frac{d}{d\tau}\left\{(z-1)\left[1-\delta(\tau)\right]\right\} = 4(G - Hz^4) . \qquad (38)$$

Substituting equation (35) into equation (38) and performing the indicated differentiation gives upon rearrangement:

$$\frac{2}{3}\,d\gamma = \frac{Hz^5 - 3Hz^4 + 2Hz^3 + Gz - G}{(G - Hz^4)^3}\,dz\ , \qquad (39)$$

in which the variables γ and z have been separated. Inasmuch as z = 1 when γ = 0, both sides of equation (39) may be integrated to obtain:

$$\frac{2}{3}\gamma = \int_1^z \frac{H\eta^5 - 3H\eta^4 + 2H\eta^3 + G\eta - G}{(G - H\eta^4)^3}\,d\eta\ . \qquad (40)$$

The integration indicated in equation (40) may be performed with moderate difficulty with the aid of a table of integrals (40), and the result verified by differentiation. The result is:

$$\begin{aligned}\gamma = {} & \frac{3(z-1)^2}{8(G - Hz^4)^2} + \frac{3z(z-1)}{16G(G - Hz^4)} \\ & + \frac{3}{32H^{1/2}G^{3/2}}\ln\frac{(H^{1/2}z^2 + G^{1/2})(H^{1/2} - G^{1/2})}{(H^{1/2}z^2 - G^{1/2})(H^{1/2} + G^{1/2})} \\ & + \frac{9}{64H^{1/4}G^{7/4}}\ln\frac{(H^{1/4}z - G^{1/4})(H^{1/4} + G^{1/4})}{(H^{1/4}z + G^{1/4})(H^{1/4} - G^{1/4})} \\ & + \frac{9}{32H^{1/4}G^{7/4}}\left[\tan^{-1}\left(\frac{H^{1/4}}{G^{1/4}}\right) - \tan^{-1}\left(\frac{H^{1/4}z}{G^{1/4}}\right)\right]\ . \qquad (41)\end{aligned}$$

For the special case where G = 0, equation (41) reduces to:

$$\tau = \frac{3}{168H^2z^8}(z^8 + 14z^2 - 36z + 21) , \qquad (42)$$

a result that may be easily derived by integrating equation (40) after first setting G = 0.

Equation (41) provides a relation between $z(\tau)$ and τ which, for known G and H, may be solved, albeit indirectly, for z as a function of τ. Thus it has been possible to obtain an approximation to the surface temperature history for the semi-infinite body. The approximate temperature at interior points of the body may be found from the following formula, developed by substituting equation (35) into equation (34):

$$t(\xi,\tau) = 1 + \frac{\left[(G - Hz^4)\xi + Hz^4 + 3z - 3 - G\right]^3}{27(z - 1)^2} . \qquad (43)$$

Equation (43) holds for $\xi \geq \delta(\tau)$; for $\xi < \delta(\tau)$, $t(\xi,\tau) = 1$. The quantity $\delta(\tau)$ may be determined from equation (35), once z has been determined.

The total heat transferred to the body from time zero to time τ is equal to the energy content of the body at time τ less its initial energy content. This gives the result:

$$Q(\tau) = A\rho c T_o L \int_{\delta(\tau)}^{1} \left[t(\xi,\tau) - 1 \right] d\xi \ . \qquad (44)$$

Substituting equation (43) into equation (44), integrating, and then substituting equation (35), leads to:

$$\frac{Q(\tau)}{A\rho c T_o L} = \frac{3(z - 1)^2}{4(G - Hz^4)} \ . \qquad (45)$$

The starting solution ceases to be valid for the problem of an infinite plate as soon as the penetration depth reaches the insulated face ($\xi = 0$). This occurs when $\delta(\tau) = 0$. Inasmuch as a different solution is applicable thereafter, it is important to know the time τ_o, the surface temperature z_o, the temperature distribution $t_o(\xi)$, and the heat transfer Q_o, valid when $\delta(\tau) = 0$, as initial conditions for the continued solution for the infinite plate. From equation (35), when $\delta(\tau) = 0$:

$$Hz_o^4 + 3z_o - (G + 3) = 0. \qquad (46)$$

This quartic equation may be solved for z_o by any of several methods, including the direct method although an iterative method is probably the most practical. The process of obtaining a general solution to equation (46) is facilitated by defining:

$$\bar{z}_o = H^{1/3} z_o \; , \tag{47}$$

$$\bar{G} = (G + 3) H^{1/3} \; . \tag{48}$$

With these definitions, equation (46) may be rewritten as:

$$\bar{z}_o^{\,4} + 3\bar{z}_o - \bar{G} = 0 \; . \tag{49}$$

In Fig. 4, $\bar{z}_o$ is plotted as a function of $\bar{G}$. From this figure, using equations (47) and (48), the value of z_o may be obtained for a given choice of G and H.

Once z_o has been obtained, the other quantities mentioned above may readily be found. These are:

$$\begin{aligned}\tau_o = {} & \frac{1}{24} + \frac{z_o}{16G} \\ & + \frac{3}{32H^{1/2}G^{3/2}} \ln \frac{(H^{1/2}z_o{}^2 + G^{1/2})(H^{1/2} - G^{1/2})}{(H^{1/2}z_o{}^2 - G^{1/2})(H^{1/2} + G^{1/2})} \\ & + \frac{9}{64H^{1/4}G^{7/4}} \ln \frac{(H^{1/4}z_o - G^{1/4})(H^{1/4} + G^{1/4})}{(H^{1/4}z_o + G^{1/4})(H^{1/4} - G^{1/4})} \\ & + \frac{9}{32H^{1/4}G^{7/4}} \left[\tan^{-1}\left(\frac{H^{1/4}}{G^{1/4}}\right) - \tan^{-1}\left(\frac{H^{1/4}z}{G^{1/4}}\right) \right] ; \end{aligned} \tag{50}$$

$$t_o(\xi) = 1 + (z_o - 1)\xi^3 \; ; \tag{51}$$

$$\frac{Q_o}{A\rho c T_o L} = \frac{z_o - 1}{4} \; . \tag{52}$$

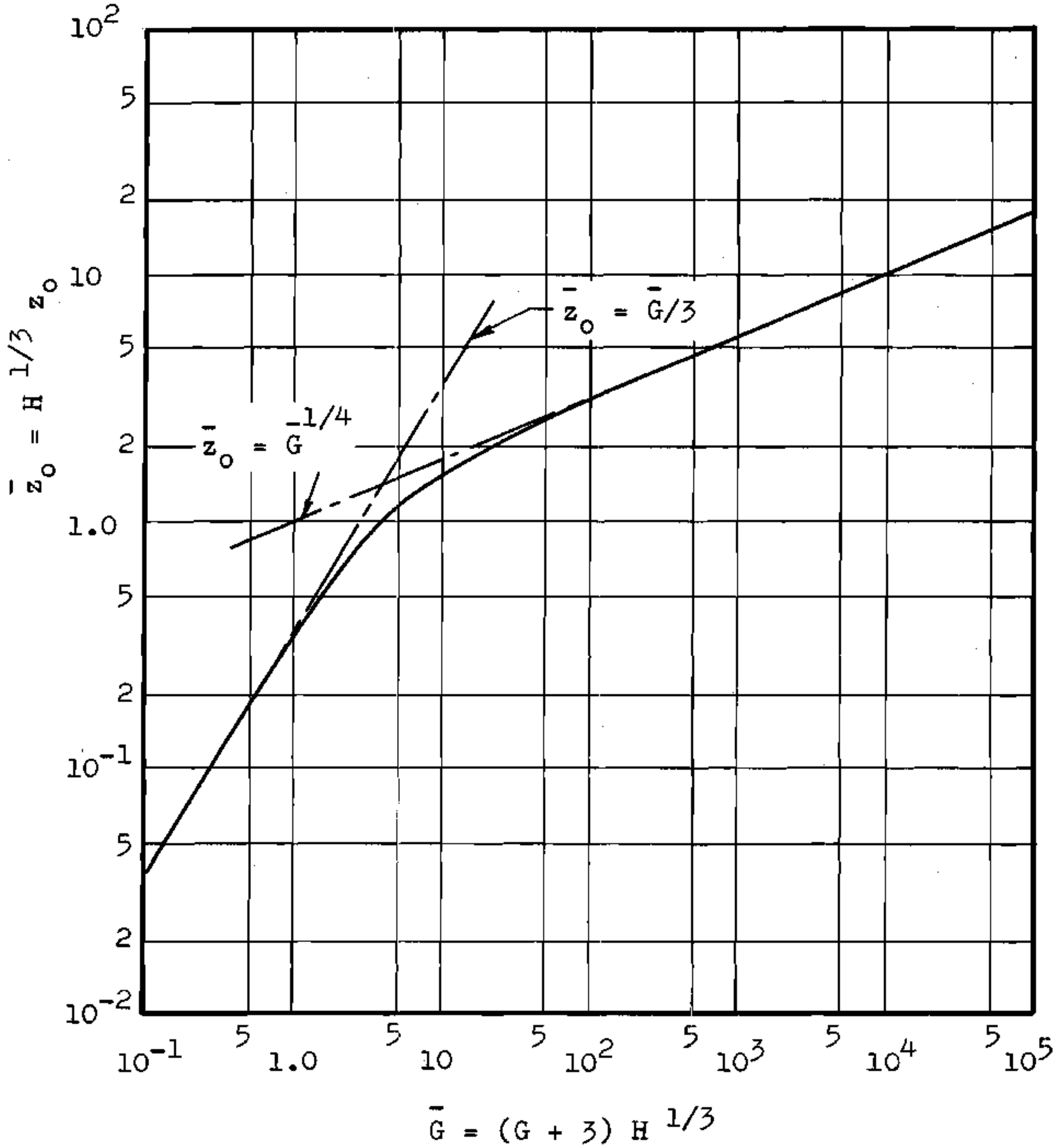


Figure 4. Solution of the Equation $\bar{z}_o^{\,4} + 3\bar{z}_o - \bar{G} = 0$

One additional quantity of interest is the following:

$$\left(\frac{dz}{d\gamma}\right)_o = \frac{6(z_o - 1)}{\frac{2}{3} Hz_o^{\,3} + 1} . \tag{53}$$

This completes the starting solution.

The Continued Solution.--Mathematically, the continued solution is as follows:

$$\frac{\partial t}{\partial \gamma} = \frac{\partial^2 t}{\partial \xi^2} , \tag{54}$$

$$t = t(\xi, \gamma),\ 0 < \xi < 1,\ \gamma \geq \gamma_o ;$$

$$t(\xi, \gamma_o) = t_o(\xi),\ 0 \leq \xi \leq 1 ; \tag{55}$$

$$\frac{\partial t}{\partial \xi}(0, \gamma) = 0, \quad \gamma \geq \gamma_o ; \tag{56}$$

$$\frac{\partial t}{\partial \xi}(1, \gamma) = G - Ht^4(1, \gamma), \quad \gamma \geq \gamma_o . \tag{57}$$

Figure 5 shows the conditions which apply during the domain of the continued problem. Making use of equation (28), equation (57) is replaced by:

$$\frac{\partial t}{\partial \xi}(1, \gamma) = G - Hz^4, \quad \gamma \geq \gamma_o . \tag{58}$$

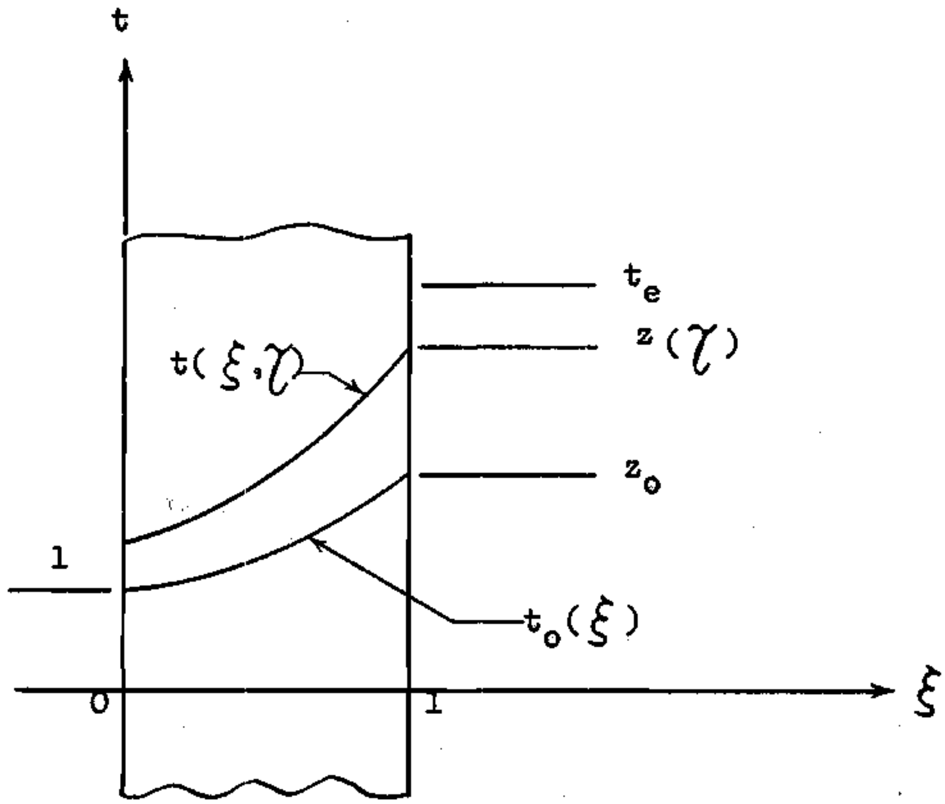


Figure 5

Section of the Infinite Plate, with Insulated Boundary

Differentiating equation (56) with respect to γ gives:

$$\frac{\partial^2 t}{\partial \xi \, \partial \gamma}(0,\gamma) = 0 \ ;$$

then differentiating equation (54) with respect to ξ gives, in conjunction with the above equation:

$$\frac{\partial^3 t}{\partial \xi^3}(0,\gamma) = 0 \ . \tag{59}$$

In a similar way, the following condition is obtained:

$$\frac{\partial^3 t}{\partial \xi^3}(1,\gamma) = -4Hz^3 \frac{dz}{d\gamma} \ . \tag{60}$$

The approximate expression for $t(\xi,\gamma)$ must satisfy as many as possible of the constraints expressed by equations (55), (56), (28), (58), (59) and (60).

The solution derived herein will be based on the following quadratic approximation to $t(\xi,\gamma)$, which automatically satisfies equations (56) and (59):

$$t(\xi,\gamma) = A_1(\gamma) + A_2(\gamma)\xi^2 . \tag{61}$$

With only two coefficients in equation (61), it will be possible to satisfy only two of the remaining constraints of equations (55), (28), (58), and (60). These coefficients will be evaluated so as to satisfy equations (28) and (58), the first order conditions at the two boundaries of the plate. In general it is preferable, in applying the heat balance integral method, to satisfy those boundary conditions having lower order derivatives in preference to those having higher order derivatives (5).

Applying equation (61) to equations (28) and (58), gives for A_1 and A_2:

$$A_1 = z - \frac{G - Hz^4}{2}, \tag{62}$$

$$A_2 = \frac{G - Hz^4}{2}. \tag{63}$$

Then $t(\xi, \tau)$ becomes:

$$t(\xi, \tau) = z - \frac{G - Hz^4}{2} + \frac{G - Hz^4}{2} \xi^2, \tag{64}$$

and from this:

$$\frac{\partial t}{\partial \tau}(\xi, \tau) = \left[1 + 2Hz^3(1 - \xi^2) \right] \frac{dz}{d\tau}. \tag{65}$$

In a manner similar to that used in obtaining equation (37), the heat balance integral is found to be:

$$\frac{d}{d\zeta}\int_0^1 t(\xi,\zeta)\, d\xi = G - Hz^4 . \tag{66}$$

Substituting equation (64) into equation (66), performing the indicated integration and differentiation, leads finally to the relation:

$$d\zeta = \frac{\frac{4}{3}Hz^3 + 1}{G - Hz^4}\, dz . \tag{67}$$

Noting that $z = z_o$ when $\zeta = \zeta_o$, one gets upon integrating equation (67):

$$\zeta - \zeta_o = \int_{z_o}^{z} \frac{\frac{4}{3}H\eta^3 + 1}{G - H\eta^4} d\eta ,$$

or, performing the integration with the aid of tables (40):

$$\zeta - \zeta_o = 1/3 \ln\left(\frac{|G - Hz_o{}^4|}{|G - Hz^4|}\right)$$

$$+ \frac{1}{4G^{3/4}H^{1/4}} \ln\left[\frac{(G^{1/4}+H^{1/4}z)(|G^{1/4}-H^{1/4}z_o|)}{(|G^{1/4}-H^{1/4}z|)(G^{1/4}+H^{1/4}z_o)}\right]$$

$$+ \frac{1}{2G^{3/4}H^{1/4}}\left[\tan^{-1}\left(\frac{H^{1/4}z}{G^{1/4}}\right) - \tan^{-1}\left(\frac{H^{1/4}z_o}{G^{1/4}}\right)\right] \tag{68}$$

If G = 0, equation (68) reduces to:

$$\tau - \tau_o = -\frac{4}{3}\ln\frac{z}{z_o} - \frac{z^3 - z_o^3}{3Hz_o^3 z^3}. \qquad (69)$$

If $\bar{t}_o(\xi)$ represents the value of $t(\xi,\tau)$ obtained from equation (64) when $\tau = \tau_o$, then:

$$\bar{t}_o(\xi) = z_o - \frac{G - Hz_o^4}{2} + \frac{G - Hz_o^4}{2}\xi^2 .$$

Using equation (46), this becomes:

$$\bar{t}_o(\xi) = \frac{3 - z_o}{2} + \frac{3}{2}(z_o - 1)\xi^2 . \qquad (70)$$

Equation (70) differs from equation (51) in comsequence of the fact that a quadratic approximation is used for the continued solution, whereas a third degree polynomial is used in the starting solution. The temperature $\bar{t}_o(0) = (3 - z_o)/2$, is less than unity by the amount $(z_o - 1)/2$. This error can perhaps be reduced by going to higher-order polynomial approximations to $t(\xi,\tau)$, such as a quartic approximation. This point is discussed in more detail later.

The energy corresponding to equation (70) is:

$$\frac{\bar{Q}_o}{A\rho c T_o L} = \int_o^1 \left[\bar{t}_o(\xi) - 1\right] d\xi = 0 .$$

This result, which is obviously in error, is another manifestation of the effect discussed in the preceding paragraph. The energy content of the plate at time γ_o is more accurately given by equation (52).

The energy content of the plate at time γ is given by:

$$\frac{Q(\gamma)}{A\rho cT_oL} = \int_o^1 \left[t(\xi,\gamma) - \bar{t}_o(\xi)\right] d\xi + \frac{z_o - 1}{4}, \qquad (71)$$

where use is made of equation (52). Substituting equation (64) into equation (71) results in:

$$\frac{Q(\gamma)}{A\rho cT_oL} = \frac{1}{3} Hz^4 + z - \frac{1}{3} G + \frac{1}{4} z_o - \frac{5}{4} . \qquad (72)$$

The solution developed in the preceding paragraphs was based on the assumption of quadratic temperature distribution. This distribution was made to satisfy the conditions imposed by equations (28), (56) (58), and (59). The constraints imposed by equations (55) and (60) could not be applied, in addition to the above constraints, to a quadratic polynomial, which has only three coefficients to be evaluated. Only by going to higher-order polynomials can all of the above constraints be applied.

An attempt is now made to obtain an approximate solution based on a quartic temperature distribution of the form:

$$t(\xi,\gamma) = A_1(\gamma) + A_2(\gamma)\xi^2 + A_3(\gamma)\xi^4 , \qquad (73)$$

which automatically satisfies equations (56) and (59). The coefficients A_1, A_2, and A_3 are to be evaluated so as to satisfy equations (28), (58), and (60), no attempt being made to satisfy equation (55). Evaluating A_1, A_2, and A_3 in terms of the surface temperature z leads to the following equation for $t(\xi,\gamma)$:

$$t(\xi,\gamma) = z - \frac{G - Hz^4}{2} - \frac{1}{6} Hz^3 \frac{dz}{d\gamma}$$

$$+ \left(\frac{1}{3} Hz^3 \frac{dz}{d\gamma} + \frac{G - Hz^4}{2} \right) \xi^2 - \frac{1}{6} Hz^3 \frac{dz}{d\gamma} \xi^4 . \qquad (74)$$

Substituting equation (74) into the heat balance integral, equation (66), integrating, differentiating, and simplifying, finally leads to the following nonlinear differential equation:

$$\frac{d^2z}{d\gamma^2} - \left(15 + \frac{45}{4Hz^3}\right) \frac{dz}{d\gamma} + \frac{3}{z}\left(\frac{dz}{d\gamma}\right)^2$$

$$= - \frac{45(G - Hz^4)}{4Hz^3} . \qquad (75)$$

Because of its nonlinearity, solving equation (75) analytically appears to be a hopeless task, and thus a numerical solution was attempted.

Equation (75) was solved numerically for one particular choice of values of G and H. It was found that this solution afforded a slightly poorer approximation to the exact solution than did the quadratic solution. In view of the fact that the quartic temperature approximation, equation (74), satisfies higher order boundary conditions than does the quadratic approximation, equation (64), this fact is quite surprising.

From equations (64) and (74), the following difference is now formed:

$$\left[t(\xi,\gamma)_{\text{quartic}} - t(\xi,\gamma)_{\text{quadratic}}\right] = -\frac{1}{6}Hz^3\frac{dz}{d\gamma}(1-\xi^2)^2 . \tag{76}$$

Also, from equations (51) and (64), evaluated when $\gamma = \gamma_o$ (or $z = z_o$), and upon using equation (46), the following difference is found:

$$\left[t(\xi,\gamma_o)_{\text{quadratic}} - t_o(\xi)\right] = -\frac{1}{2}(1-\xi)(z_o - 1)(1 - \xi + 2\xi^2) . \tag{77}$$

From equations (21) and (22):

$$z_e \equiv \frac{T_e}{T_o} = \left(\frac{G}{H}\right)^{1/4} . \tag{78}$$

When $G > H$: $z_e > 1$, $1 < z_o < z_e$, and $dz/d\gamma > 0$. Thus, the differences expressed by equations (76) and (77) are each negative for $0 \leq \xi < 1$, since $1 - \xi + 2\xi^2 > 0$ in this range. Thus it is seen that the quartic approximation at $\gamma = \gamma_o$ differs from $t_o(\xi)$ by a greater amount than does the quadratic approximation. This same conclusion is also arrived at when $G < H$. As pointed out previously, this discrepancy between $t(\xi, \gamma_o)$ from the continued solution and $t_o(\xi)$ from the starting solution, is an inherent defect in the approximation technique. However, it now appears that in the process of satisfying higher order boundary conditions, this discrepancy becomes more pronounced. This suggests that in order to obtain a better approximation than the quadratic solution, attention must be given to the initial condition, equation (55).

Before any attempt is made to satisfy the initial condition, equation (55), two observations should be made. First, some discrepancy between $t(\xi, \gamma_o)$ from the continued solution and $t_o(\xi)$ is inevitable, owing to the different forms of $t(\xi, \gamma)$ in the starting and continued solutions, made necessary by the different boundary conditions which apply to the two solutions. This discrepancy can best be minimized by equating the energies represented by $t(\xi, \tau_o)$ and $t_o(\xi)$. This gives:

$$\int_o^1 t_o(\xi)d\xi = \int_o^1 t(\xi, \tau_o)d\xi \quad . \tag{79}$$

Thus the initial condition will be said to be satisfied if equation (79) is satisfied.

The second observation is as follows. To satisfy equation (79), either a boundary condition must be omitted, or a higher degree polynomial approximation to $t(\xi, \tau)$ must be employed. Since equation (79) applies only at time τ_o, one of the coefficients in the approximating polynomial can be determined only at $\tau = \tau_o$, its time variation being left undetermined by this procedure. Unless the time variation of this coefficient be determined in such a way as to satisfy, or partially satisfy, an additional boundary condition, difficulties will be encountered. The reason for this is that a high degree polynomial, without sufficient constraint, may show maxima, minima, and points of inflection which do not appear in the exact solution. Fig. 6 shows such a situation, wherein two approximations, one of degree higher than the other, both satisfy the same boundary conditions.

The conclusion now becomes evident. In order to derive a quartic or quintic solution which is more accurate than the quadratic approximation, it will be necessary to

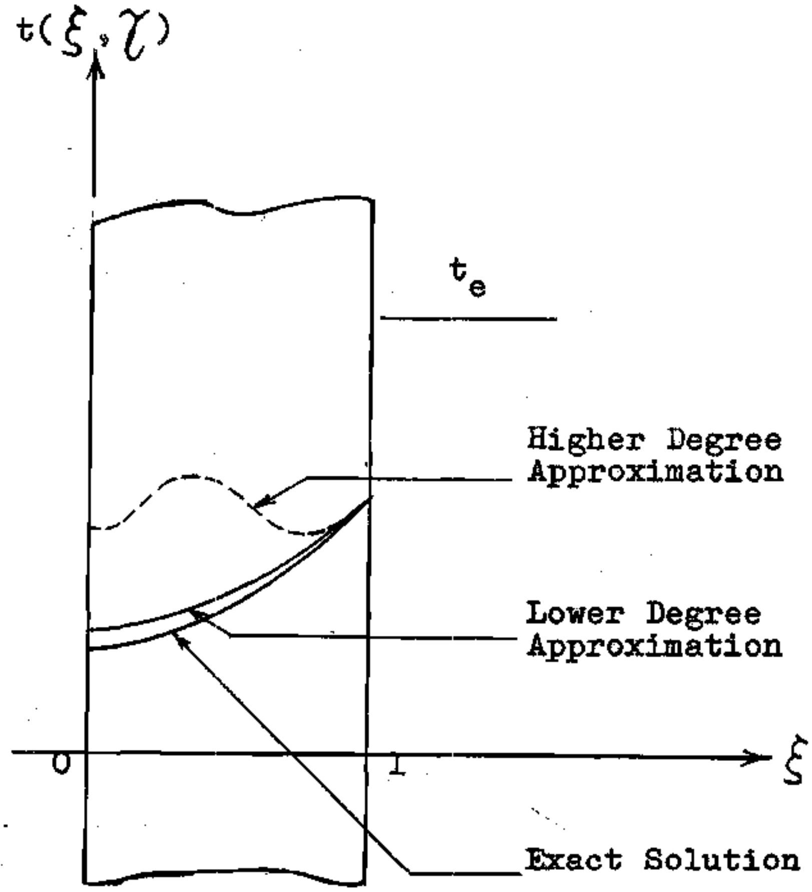


Figure 6

Two Approximate Solutions

devise a scheme whereby one of the coefficients in the approximate temperature distribution can be made to satisfy equation (79) when $\gamma = \gamma_0$, and in some sense to satisfy an additional boundary condition when $\gamma > \gamma_0$. Such a scheme is not presented in the present work.

CHAPTER III

NUMERICAL SOLUTION

Outline of Procedure.--In this chapter an algorithm is derived for solving numerically the problem expressed by equations (17)-(20). Also, an attempt is made to predict the error inherent in this algorithm. However, in view of the complicated boundary condition, equation (20), an exact error bound is not obtained. The question of stability of the numerical algorithm is considered.

In performing numerical solutions of this type, it is customary to divide the plate into a grid consisting of a finite number of grid points. Then, starting from an initial temperature distribution, the temperature at each grid point at time $\tau + \Delta\tau$ is computed from the known temperatures at time τ, $\tau - \Delta\tau$, etc. This process utilizes finite-difference equations derived from the differential equation and boundary conditions which apply to the problem. This process is repeated until the temperature in the plate approaches the steady-state temperature as closely as desired. In this way the transient temperature distribution in the plate is obtained.

Although the starting temperature of unity, as contained in equation (18), is applicable to this problem, actually starting the solution at this point has two disadvantages. These arise from the fact that at the point $\xi = 1$ at time $\tau = 0$, there exist discontinuites in the temperature derivatives of all orders, and moreover, these derivatives are large in magnitude at points in the vicinity of the surface at small time values. The first disadvantage is that it is difficult to determine suitable starting values of temperature at times $-\Delta\tau$, $-2\Delta\tau$, etc. at the point $\xi = 1$, should these be needed in the numerical algorithm. Secondly, no matter what algorithm is employed in the numerical solution, errors made near the outset of the solution are relatively much larger than those made during the same time interval later on in the solution. For these reasons, a more satisfactory initial temperature distribution is sought.

The solution of Jaeger (2), alluded to previously, for the semi-infinite body, initially at a uniform temperature throughout, which suddenly begins to exchange energy at its surface according to a power law, affords an ideal starting solution. Until the temperature at the center of the plate begins to change appreciably, the plate behaves essentially as a semi-infinite body, to which Jaeger's solution applies. Thus the temperature distribution in the plate as determined from Jaeger's solution at time τ_o is to be used as the starting temperature distribution in the numerical solution. The time τ_o is that time at which the temperature at $\xi = 0$, as determined from Jaeger's solution, differs from unity by an amount not greater than half the acceptable maximum error in the numerical solution. This value of τ_0 is approximately equal to the value of τ_0 determined in connection with the approximate solution of Chapter II. The temperature distributions at times $\tau_0 - \Delta\tau$, $\tau_0 - 2\Delta\tau$, etc., may also be readily obtained from Jaeger's solution.

The Numerical Algorithm.--In this section an algorithm is developed for the numerical solution of the mathematical problem expressed by equations (17)-(20). It is assumed that the initial temperature distributions at times τ_0, $\tau_0 - \Delta\tau$, etc., are known. Except for the treatment of the boundary condition at $\xi = 1$, the solution developed herein is essentially that presented by Milne (41).

Let the plate $0 \leq \xi \leq 1$ be divided into M segments, giving M + 1 grid points. Let:

$$\xi = mh \ , \quad m = 0, 1, 2, \ldots, M \ , \tag{80}$$

$$\tau = \tau_0 + nk \ , \quad n = 0, 1, 2, \ldots, -1, -2, \ldots \ , \tag{81}$$

in which h is the space interval 1/M, and k is the time interval yet to be determined. Also, the following notation is used:

$$t_{m,n} = t(mh, \tau_0 + nk) \ .$$

With the above notation, the mathematical problem of equations (17)-(20) becomes:

$$\frac{\partial t_{m,n}}{\partial \tau} = \frac{\partial^2 t_{m,n}}{\partial \xi^2}, \tag{82}$$

$$m = 0, 1, 2, \ldots, M, \quad n = 0, 1, 2, \ldots ;$$

$$\frac{\partial t_{o,n}}{\partial \xi} = 0, \quad n = 0, 1, 2, \ldots ; \tag{83}$$

$$\frac{\partial t_{M,n}}{\partial \xi} = G - H\, t_{M,n}^4, \quad n = 0, 1, 2, \ldots . \tag{84}$$

In addition, $t_{m,0}$, $m = 0, 1, 2, \ldots, M$ is known, as is $t_{m,-1}$, $t_{m,-2}$, etc., as needed. The following equations, obtained from equation (82) will prove useful:

$$\frac{\partial^2 t_{m,n}}{\partial \tau^2} = \frac{\partial^3 t_{m,n}}{\partial \tau \partial \xi^2} = \frac{\partial^4 t_{m,n}}{\partial \xi^4}, \tag{85}$$

$$\frac{\partial^3 t_{m,n}}{\partial \tau^3} = \frac{\partial^5 t_{m,n}}{\partial \tau \partial \xi^4} = \frac{\partial^6 t_{m,n}}{\partial \xi^6}. \tag{86}$$

In this derivation it is assumed that the derivatives of $t_{m,n}$ of all orders are continuous throughout the range $0 \leq \xi \leq 1$, $\tau \geq \tau_o > 0$. As a result, a Taylor series expansion for temperature may be written about any point (ξ, τ) within the range mentioned. Writing Taylor series in ξ for $m = 1, 2, \ldots, M-1$, $n = 0, 1, 2, \ldots$:

$$t_{m+1,n} = t_{m,n} + h\frac{\partial t_{m,n}}{\partial \xi} + \frac{h^2}{2}\frac{\partial^2 t_{m,n}}{\partial \xi^2} + \frac{h^3}{6}\frac{\partial^3 t_{m,n}}{\partial \xi^3} + \frac{h^4}{24}\frac{\partial^4 t_{m,n}}{\partial \xi^4} + \frac{h^5}{120}\frac{\partial^5 t_{m,n}}{\partial \xi^5} + \frac{h^6}{720}\frac{\partial^6 t_{m+\zeta_1,n}}{\partial \xi^6}, \qquad (87)$$

and:

$$t_{m-1,n} = t_{m,n} - h\frac{\partial t_{m,n}}{\partial \xi} + \frac{h^2}{2}\frac{\partial^2 t_{m,n}}{\partial \xi^2} - \frac{h^3}{6}\frac{\partial^3 t_{m,n}}{\partial \xi^3} + \frac{h^4}{24}\frac{\partial^4 t_{m,n}}{\partial \xi^4} - \frac{h^5}{120}\frac{\partial^5 t_{m,n}}{\partial \xi^5} + \frac{h^6}{720}\frac{\partial^6 t_{m-\zeta_2,n}}{\partial \xi^6}, \qquad (88)$$

where $0 \leq \zeta_1, \zeta_2 \leq 1$. Adding equation (87) to equation (88) gives, upon rearrangement:

$$\frac{\partial^2 t_{m,n}}{\partial \xi^2} = \frac{t_{m+1,n} - 2t_{m,n} + t_{m-1,n}}{h^2} - \frac{h^2}{12}\frac{\partial^4 t_{m,n}}{\partial \xi^4} - \frac{h^4}{360}\frac{\partial^6 t_{m+\zeta_3,n}}{\partial \xi^6}, \qquad (89)$$

where $-1 \leq \zeta_3 \leq 1$.

A Taylor series may also be written in γ, which when rearranged gives:

$$\frac{\partial t_{m,n}}{\partial \gamma} = \frac{t_{m,n+1} - t_{m,n}}{k} - \frac{k}{2}\frac{\partial^2 t_{m,n}}{\partial \gamma^2} - \frac{k^2}{6}\frac{\partial^3 t_{m,n+\mu_1}}{\partial \gamma^3}, \qquad (90)$$

in which $m = 1, 2, \ldots, M-1$, $n = 0, 1, 2, \ldots$, and $0 \leq \mu_1 \leq 1$.

Substituting equations (89) and (90) into equation (82), and utilizing equations (85) and (86) leads to:

$$t_{m,n+1} = \frac{k}{h^2} t_{m+1,n} + \left(1 - \frac{2k}{h^2}\right) t_{m,n} + \frac{k}{h^2} t_{m-1,n} + \left(\frac{k^2}{2} - \frac{h^2 k}{12}\right)\frac{\partial^4 t_{m,n}}{\partial \xi^4} + \frac{k^3}{6}\frac{\partial^6 t_{m,n+\mu_1}}{\partial \xi^6} - \frac{h^4 k}{360}\frac{\partial^6 t_{m+\zeta_3,n}}{\partial \xi^6}. \qquad (91)$$

If:

$$k = h^2/6, \qquad (92)$$

then equation (91) becomes:

$$t_{m,n+1} = \frac{t_{m+1,n} + 4t_{m,n} + t_{m-1,n}}{6} + P_1 , \qquad (93)$$

in which:

$$P_1 = \frac{h^6}{1296} \frac{\partial^6 t_{m,n+\mu_1}}{\partial \xi^6} - \frac{h^6}{2160} \frac{\partial^6 t_{m+\zeta_3,n}}{\partial \xi^6} . \qquad (94)$$

It is recalled that equation (93) is valid for $m = 1,2,\ldots, M-1$, $n = 0, 1, 2, \ldots$, and that in equation (94) $0 \leq \mu_1 \leq 1$, $-1 \leq \zeta_3 \leq 1$.

Equation (89) may be readily modified to apply when $m = 0$, $n = 0, 1, 2, \ldots$. By defining a fictitious $m = -1$, equation (83) may be replaced by the condition:

$$t_{-1,n} = t_{1,n} , \quad n = 0, 1, 2, \ldots . \qquad (95)$$

Then, utilizing equation (95), equation (93) becomes:

$$t_{o,n+1} = \frac{t_{1,n} + 2t_{o,n}}{3} + P_1 . \qquad (96)$$

In equation (96), P_1 is given by equation (94), with $m = 0$.

In the numerical algorithm, equations (93) and (96) are to be used, after dropping the term P_1, which then becomes the truncation error for a single time step. It is seen that P_1 is of the order h^6. The corresponding relation for the grid point at the boundary m = M ($\xi = 1$) is derived in Appendix A. This equation has the truncation error term P_2', which is also of the order h^6. After omitting the truncation error terms, the numerical algorithm is now summarized:

$$t_{m,n+1} = \frac{t_{m+1,n} + 4t_{m,n} + t_{m-1,n}}{6} , \tag{97}$$

$$m = 1, 2, \ldots, M-1, \quad n = 0, 1, 2, \ldots ;$$

$$t_{o,n+1} = \frac{t_{1,n} + 2t_{o,n}}{3} , \quad n = 0, 1, 2, \ldots ; \tag{98}$$

$$t_{M,n+1} = D_1 + D_2\, t^2_{M,n+1} , \ n = 0, 1, 2, \ldots . \tag{99}$$

The following defined terms are utilized:

$$D_o = 3 + \frac{16}{5} Hh\, t^2_{M,n}(t_{M,n} - \frac{9}{16} t_{M,n-1}) \quad ; \tag{100}$$

$$D_1 = \frac{1}{D_o}\left(t_{M-1,n} + hG + t_{M,n}\left\{2 + \frac{4}{5} Hh\, t_{M,n}\left[-\frac{9}{8} t^2_{M,n-1} + t_{M,n}(t_{M,n-1} + \frac{7}{4}t_{M,n})\right]\right\}\right) ; \tag{101}$$

$$D_2 = -\frac{9}{10} hH\, t^2_{M,n}/D_o . \tag{102}$$

The required initial data are $t_{m,0}$ for $m = 0, 1, 2, \ldots, M$ and also $t_{M,-1}$. Once the space interval $h = 1/M$ has been selected, the time interval k is automatically fixed by virtue of equation (92).

The numerical algorithm expressed by equations (97)-(102) was utilized in writing a computer program for use on the Burroughs 220 computer with algebraic complier. This program is reproduced in Appendix B.

Error and Stability Analysis.--In this section an analysis is made of the maximum error to be expected in the execution of the numerical algorithm expressed by equations (97)-(102). In this connection there are two major considerations to be taken into account. The first consideration is the question of stability; that is, the condition whereby small errors do not magnify as the solution proceeds. Stability is demonstrated in the case of equations (97) and (98), but is simply assumed for the case of equation (99), although a nonrigorous argument is given in support of such an assumption. The second consideration is that of the maximum accumulation of truncation error by the time the numerical solution has been completed. In estimating the magnitude of this accumulation of truncation errors, it is assumed that all truncation errors are of their greatest magnitude throughout and in the same direction, and that none of them are attenuated as the numerical solution proceeds. Even with these assumptions, it is impossible to obtain an exact error estimate, since the sixth order derivatives in the expressions for P_1 and P_2 (equations (94) and (A11) of Appendix A) cannot be evaluated *a priori*. However, these truncation error terms are assumed to be bounded, as follows:

$$\left|P_1\right| , \left|P_2'\right| < Ph^6 . \qquad (103)$$

The stability of equations (97) and (98) is established in Appendix C. No attempt is made to prove stability of equation (99), in view of its extreme complexity and iterative nature. However, an argument which indicates that equation (99) is stable, is as follows: According to Dusinberre (42), stability occurs as a general rule if none of the coefficients $t_{m+1,n}$, $t_{m,n}$, or $t_{m-1,n}$ on the right side of the difference equation are negative. With reference to equation (A10) in Appendix A, it is seen, upon proper rearrangement of this equation, that the coefficients of $t_{M,n}$, $t_{M-1,n}$, $t_{M,n-1}$ are all positive on the right hand side of the equation. Also the coefficient of $t_{M,n+1}$ is positive on the left hand side of the equation. On the strength of the above argument, it is assumed that equation (99) is stable. If not, this fact should at least be evident from the numerical solution itself.

The accumulation of truncation error is next considered. In Appendix D it is shown, subject to certain approximations, that the truncation error for a single time step is bounded by:

$$E_{n+1} - E_n \leq Ph^6 , \tag{104}$$

where E_n represents an upper bound on the absolute truncation error after n time steps. Thus it follows immediately that:

$$E_n \leq E_o + nPh^6 . \tag{105}$$

Since:

$$n = \frac{\tau - \tau_o}{k} = \frac{6(\tau - \tau_o)}{h^2} , \tag{106}$$

then:

$$E(\tau) \leq E_o + 6P(\tau - \tau_o)h^4 . \tag{107}$$

Equation (107) then affords an error estimate after time τ has elapsed. However, the value of P is unknown a priori, and in order to predict the error, its value must be estimated in some manner.

There are three reasons why the errors should turn out considerably less than predicted by equation (107). First, the errors made early in the solution may attenuate somewhat toward the end of the solution, as seen by equation (C19) in Appendix C. Second, not all the errors are in the same direction, and some of them are thus cancelled.

Third, the sixth-order derivatives in the expressions for P_1 and P_2 do not assume their maximum magnitude over the entire range of integration. Taking all these things into consideration, it is felt that the error estimate of equation (107) is a conservative one,

CHAPTER IV

EXPERIMENTAL SOLUTION

<u>Experimental Apparatus</u>.--An experimental solution was obtained in connection with the problem expressed in equations (17)-(20). This experiment was designed with two purposes in mind. First, it was desired to confirm the results of the approximate and numerical analyses. Second, it was desired to estimate the effectiveness of an experimental approach in cases of the generic problem not readily amenable to analytical or numerical solution.

Figure 7 shows a schematic diagram of the main portion of the experimental apparatus. An asbestos sample was placed inside a vacuum chamber, whose walls were cooled by water circulating through the surrounding jacket. Two heaters, made by stringing chromel "A" wires between mica supports, were placed opposite the faces of the sample. Three thermocouples were imbedded inside the sample near its two faces and its center. The thermocouples at the faces of the sample were imbedded slightly within the sample. A radiation shield (not shown) was placed adjacent to the edges of the sample.

Additional equipment not shown in Figure 7 consisted of a multipoint recording potentiometer, used to record the

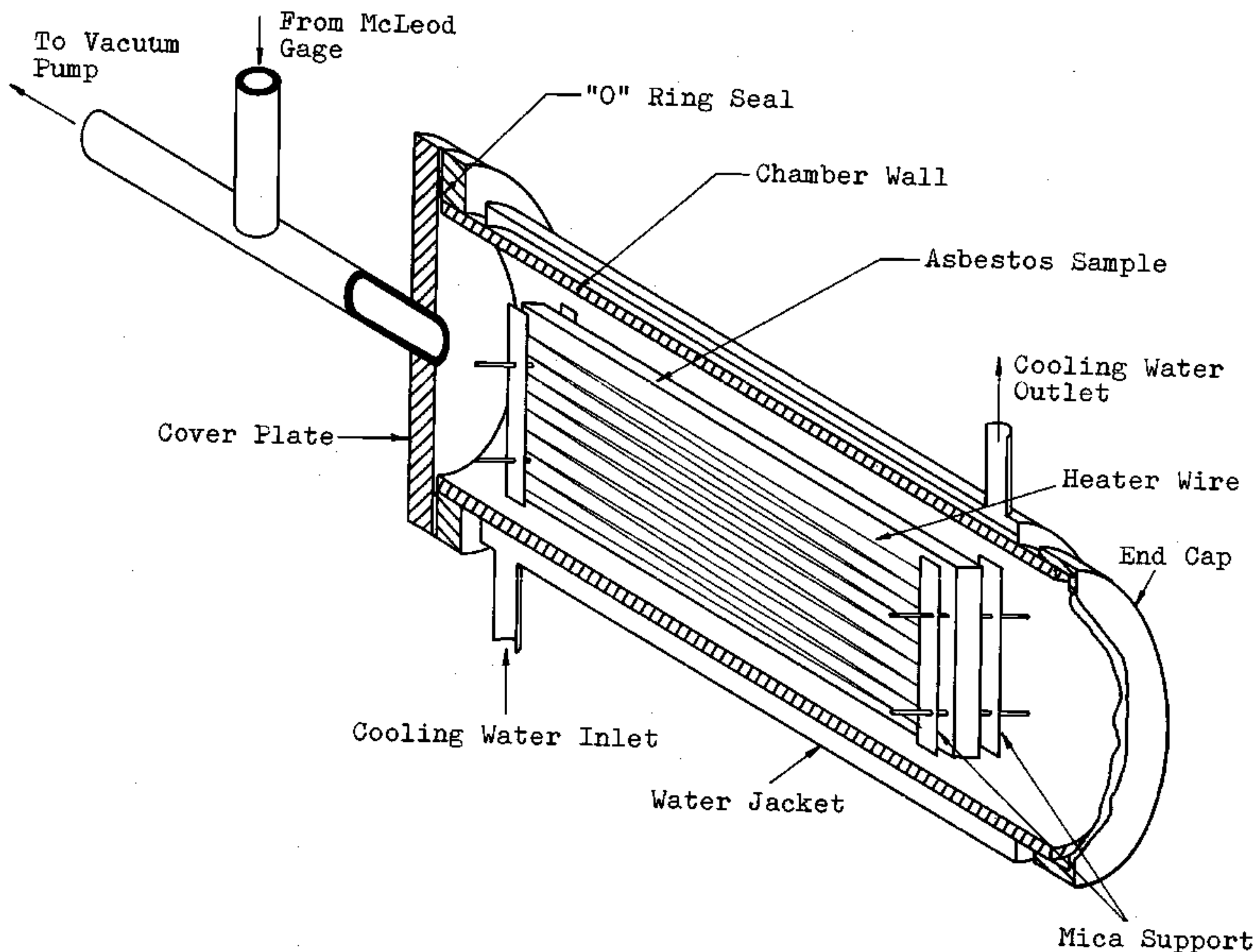


Figure 7. Schematic Diagram of Vacuum Chamber and Sample

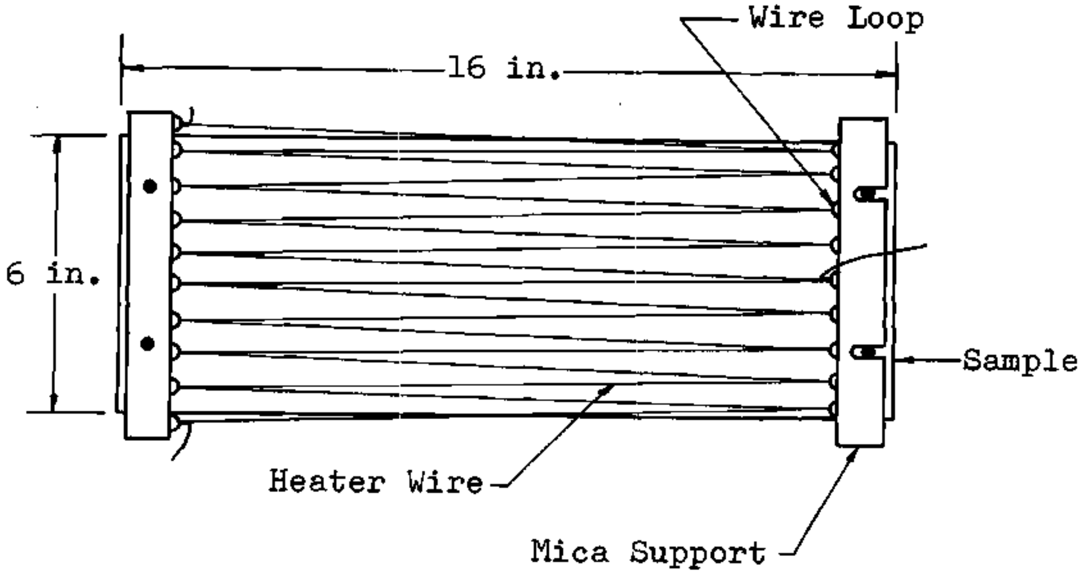
voltage output from the thermocouples imbedded in the sample as well as thermocouples attached to the inlet and outlet cooling water tubes, two variable transformers for regulating the voltage supply to the heaters, a vacuum pump and a McLeod gage used for measuring the pressure within the systhem. The thermocouple leads and heater leads were taken through the cover plate by use of special ceramic plugs available commercially for such purposes. The vacuum chamber and McLeod gage were connected to the vacuum pump by means of special vacuum hose and copper tubing.

Following a suggestion given by Peck, et al. (44), the inside surface of the vacuum chamber was coated with "glyptal" lacquer, in order to reduce the danger of leaks into the system. The cover plate was sealed to the chamber by means of an "O" ring seal.

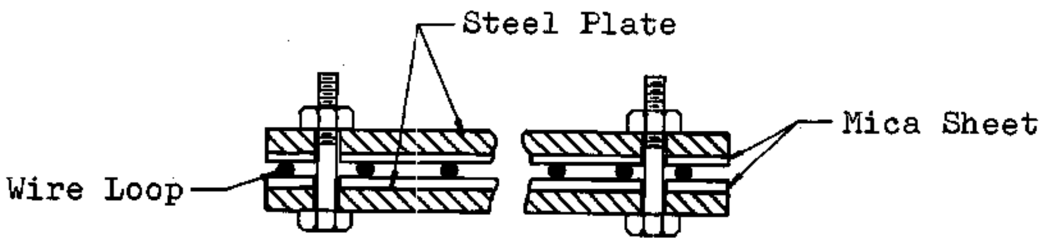
Figure 8 shows a view of the face of the sample, together with the heater and details of construction of the mica supports.

Specifications of the experimental apparatus are given in Appendix E.

Experimental Procedure.--The experimental procedure was as follows. The vacuum pump was turned on and the system reduced to a suitable pressure. The heaters were turned on and the sample brought up to a uniform temperature, which was to be the initial temperature for the run. Cooling water was circulated through the jacket.



Face of Sample



Detail of Mica Support

Figure 8. Detail of Heater Construction

After the system had reached steady state, the experimental run was started. To start the run, the heaters were switched off and the various temperatures recorded as a function of time. The temperatures recorded were the following: temperature at the center of the sample, temperatures at the two faces of the sample, inlet cooling water temperature, and outlet cooling water temperature. The run was terminated after the difference between the sample temperature and the cooling water temperature was reduced to five per cent of its initial value.

The recorded data was reduced to nondimensional form for comparison with the approximate and numerical solutions. In order to do this, it was necessary to measure the values of the shape factor combination $\sigma\mathcal{F}L/k$ and of the diffusivity α. These quantities were measured on the experimental equipment itself; the procedure for their measurement is discussed in subsequent sections. In addition, a correction for deviation from a step-function boundary condition was applied to the data, as is also discussed in a subsequent section.

Shape Factor Measurement.--The combination of properties $\sigma\mathcal{F}L/k$ was measured for both faces of the sample, with the sample in place in the experimental apparatus. The basic procedure was as follows. The heater opposite the face of the sample for which $\sigma\mathcal{F}L/k$ was to be measured was turned off, and the other heater turned on to a suitable voltage. After steady state had been achieved, readings of the surface temperatures and of the center temperature of the sample were taken. From these readings, the combination $\mathcal{F}L/k$ was computed.

Under the conditions outlined above, there is a steady heat flux through the sample, which is dissipated by radiation from the cold face to the vacuum chamber walls. Writing an energy balance about the cold face (the face of the sample for which $\mathcal{F}L/k$ is being measured) gives:

$$\frac{\mathcal{F}L}{k} = \frac{T_h - T_c}{2\sigma(T_c^{\,4} - T_e^{\,4})} \quad . \tag{108}$$

A difficulty was encountered in applying equation (108), since it was desired to measure $\mathcal{F}L/k$ for temperatures ranging from T_e to the maximum temperature which could be obtained in the system. With T_h equal to the maximum temperature obtainable in the system, T_c was found to be considerably less than T_h, such that $\mathcal{F}L/k$ could be evaluated only over about half the range indicated. To measure $\mathcal{F}L/k$ at higher temperatures, the heater opposite the cold face was turned on, but to a lower voltage than the heater opposite the hot face. Doing this effectively changes the environmental temperature opposite the cold face, thus reducing $(T_h - T_c)$. Upon denoting this new effective environmental temperature by T_e', equation (108) becomes:

$$\frac{\mathcal{F} L}{k} = \frac{T_h - T_c}{2\sigma(T_c{}^4 - T_e{}'^4)} \ . \tag{109}$$

To measure T_e', the higher heater voltage was reduced to that of the lower, and the sample allowed to reach steady state. When this had occurred, the sample temperature was that of the effective environmental temperature T_e'.

Heater Design.--It is the purpose of this section to indicate the method of designing the heaters in order to maintain the sample at a given steady temperature. It is assumed that half the energy radiated by the heater impinges on the surface of the sample, while the other half strikes the vacuum chamber walls directly. All the energy incident on the sample surface is, in steady state, either reflected or reradiated to the vacuum chamber walls. Thus sufficient energy from the heater must strike the sample surface to provide for radiant exchange between a like sample having a black surface and the vacuum chamber walls. Let $\mathcal{F}'$ denote the gray body shape factor between the sample surface and the walls, under the assumption that the sample surface is black (ϵ = 1). Then:

$$q_w = 2\sigma A_s \mathcal{F}' \ (T_o{}^4 - T_e{}^4) \ , \tag{110}$$

where T_o here represents the steady, uniform temperature at which the sample is to be maintained.

Let P_w be the watt equivalent to q_w, and thus the electrical power input to the heater. Then the heater current and voltage are given by:

$$I = \sqrt{P_w/R} \ , \tag{111}$$

$$V = \sqrt{P_w R} \ . \tag{112}$$

The complete heater design must take into consideration the supply voltage and maximum current, and the resistance per foot of heater wire. The heater design will be pursued no further; however, specifications for the heaters used are included in Appendix E.

The energy given off by the heater wire at a uniform temperature T_w is:

$$q_w = \sigma \epsilon_w A_w T_w^{\ 4} \ . \tag{113}$$

Equating this to equation (110) results in:

$$T_w = \left[K_1 (T_o^{\ 4} - T_e^{\ 4}) \right]^{1/4} , \tag{114}$$

where:

$$K_1 = \frac{2A_s \mathcal{F}'}{\epsilon_w A_w} \ . \tag{115}$$

Thus the heater wire temperature necessary to establish a steady-state sample temperature T_o, is given by equations (114) and (115).

Correction for Deviation from Step-Function Boundary Condition.--As soon as the current is shut off from the heater wires, at the beginning of the experimental run, they cool rapidly, soon reaching an equilibrium temperature intermediate between that of the sample surface and that of the wall. Although this cooling process is assumed to be instantaneous in developing the approximate and numerical solutions, in practice the process requires a finite time to complete. To compensate for this, the experimental run was assumed to have started a short time after the heaters were shut off. The amount of this time lag, denoted by $\bar{\theta}$, is a correction which was applied to the experimental data. The magnitude of $\bar{\theta}$ is such that the energy dissipated from the wire after shutting off the current is the same in the assumed step-function boundary condition as in the actual case.

Let T_w be the heater wire temperature and let its initial value before shutting off the current be denoted by T_{wo}. The total energy dissipated by the wire after the current is shut off is given by:

$$E = C_w \rho_w V_w (T_{wo} - T_e) , \qquad (116)$$

where $c_w \rho_w V_w$ is the total heat capacity of the wire. The correction $\bar{\theta}$ is to be such that:

$$E = \sigma \epsilon_w A_w \left(T_{wo}^{\,4} - T_e^{\,4} \right) \bar{\theta} \ . \qquad (117)$$

Equating the right sides of equations (116) and (117) results in:

$$\bar{\theta} = K_2 \frac{T_{wo} - T_e}{T_{wo}^{\,4} - T_e^{\,4}} \ , \qquad (118)$$

in which:

$$K_2 = \frac{\rho_w c_w V_w}{\sigma \epsilon_w A_w} \ . \qquad (119)$$

The time correction $\bar{\theta}$, as determined from equations (118) and (119), was subtracted from the observed time readings in the experimental data.

Diffusivity Measurement.--To measure the diffusivity α of the sample, the following procedure was used. First the sample was brought up to a steady temperature T_1, slightly lower than the temperature for which α was desired. Then the heater voltage was suddenly increased sufficiently to cause an ultimate rise in temperature of the sample to T_2. The variation of temperature at the center of the sample during the initial part of this heating process provided the basis for determining α.

The temperature rise $(T_2 - T_1)$ is sufficiently small--approximately 40 to 80°F.--such that the radiation boundary condition may be replaced with good accuracy by a linear boundary condition, if properly chosen. Let T_3 be a temperature such that $(T_3 - T_1) \approx (T_2 - T_1)/3$, and let T_4 be the temperature of the center of the plate at time θ_α when the surface temperature is T_3. Replace the actual fourth-power boundary condition:

$$\frac{\partial T}{\partial x}(L,\theta) = \frac{\sigma \mathcal{F}}{L}\left[T_2{}^4 - T^4(L,\theta) \right] \quad , \qquad (120)$$

by the linear boundary condition:

$$\frac{\partial T}{\partial x}(L,\theta) = b\left[T_2 - T(L,\theta) \right] \quad . \qquad (121)$$

Let equation (120) and (121) be identical at the temperature $(T_1 + T_3)/2$. This requirements leads to the results:

$$bL = \frac{1}{8}\sigma\left(\frac{\mathcal{F}L}{k}\right)\frac{16T_2{}^4 - (T_1+T_3)^4}{2T_2 - T_1 - T_3} \quad . \qquad (122)$$

The use of the boundary condition of equation (121), where b is given by equation (122), leads to a well-known convection problem, whose solution is (45):

$$\frac{T_4 - T_2}{T_1 - T_2} = \sum_{n=1}^{\infty} e^{-\delta_n^2 \frac{\alpha \theta_\alpha}{L^2}} \frac{2 \sin \delta_n}{\delta_n + \frac{1}{2} \sin 2\delta_n} . \quad (123)$$

In equation (123), θ_α is the time at which the center temperature is T_4. The term δ_n in equation (123) is the n'th root of the equation:

$$\text{ctn } \delta_n = \delta_n / bL . \quad (124)$$

From equations (122)-(124) and the observed values of T_1, T_2, T_3, and T_4, values of α at $(T_1 + T_4)/2$ were computed. Measurements of α were made over the complete range of temperature which could be obtained in the system.

The effect of deviation from a step function boundary condition, discussed in the preceding section as applied to the experimental run, also applies to the procedure for measuring α. Instead of correcting for this effect by applying a time correction as was done for the experimental run, a different form of correction was chosen for the process of measuring α. This effect was compensated for by overshooting the voltage increase for a short time, then cutting back to the correct final voltage, in such a way that the energy dissipated by the wire was the same as in the step-function case.

Let v_1 be the voltage corresponding to the steady sample temperature T_1, v_2 correspond to T_2, and v' be the maximum heater voltage which is held for a brief period θ_1 after the voltage is first increased. The energy balance gives:

$$3.413\ \frac{v'^2 - v_2^2}{R}\ \theta_1 = c_w \rho_w V_w (T_{w2} - T_{w1})\ , \qquad (125)$$

where T_{w1} and T_{w2} are the heater wire temperatures corresponding to the steady sample temperatures T_1 and T_2 respectively. From equation (125) and foregoing equations, the value of v' may be computed for an arbitrarily chosen value of θ_1.

<u>Effect of Ambient Pressure</u>.--In this section the effect of an ambient pressure greater than zero is studied. With an ambient medium present, conduction and convection effects influence the boundary condition, which would otherwise be that of pure radiation, and in turn influence the temperature within the plate.

To study the effect of such conduction and convection, an additional term is added to equation (9), with temperature now being denoted by $\widetilde{T}$:

$$\frac{\partial \widetilde{T}}{\partial x}(L,\theta) = \frac{\sigma \mathcal{F}}{k}\left[T_e^4 - \widetilde{T}^4(L,\theta)\right] + \frac{C}{k}\left[T_e - \widetilde{T}(L,\theta)\right] \ . \qquad (126)$$

In equation (126), C is the conductive or convective coefficient, and is assumed to be constant at a particular pressure for purposes of this analysis. In nondimensional form, equation (126) becomes:

$$\frac{\partial \tilde{t}}{\partial \xi}(1,\gamma) = G - H\tilde{t}^4(1,\gamma) + J - K\,\tilde{t}(1,\gamma) \ , \qquad (127)$$

in which:

$$J = CLT_e/kT_o \ , \qquad (128)$$

$$K = CL/k \qquad (129)$$

For fixed geometry and temperature of the ambient fluid, the coefficient C in equation (126) varies with pressure in a manner shown qualitatively in Fig. 9 (46). At very low pressures, the curve for C as a function of p, curve I in Fig. 9, approaches an asymptote given by (46):

$$C = a_1 p \sqrt{\frac{R_g}{T}} \ , \qquad (130)$$

where:

$$a_1 = 17.1 \ \frac{lb_m^{1/2}\ Btu}{hr\ ft^{1/2}\ lb_f^{3/2}} \ . \qquad (131)$$

Since C is assumed to be constant, it should be evaluated at some mean temperature of the ambient gas. As the pressure is increased, the coefficient C approaches a constant

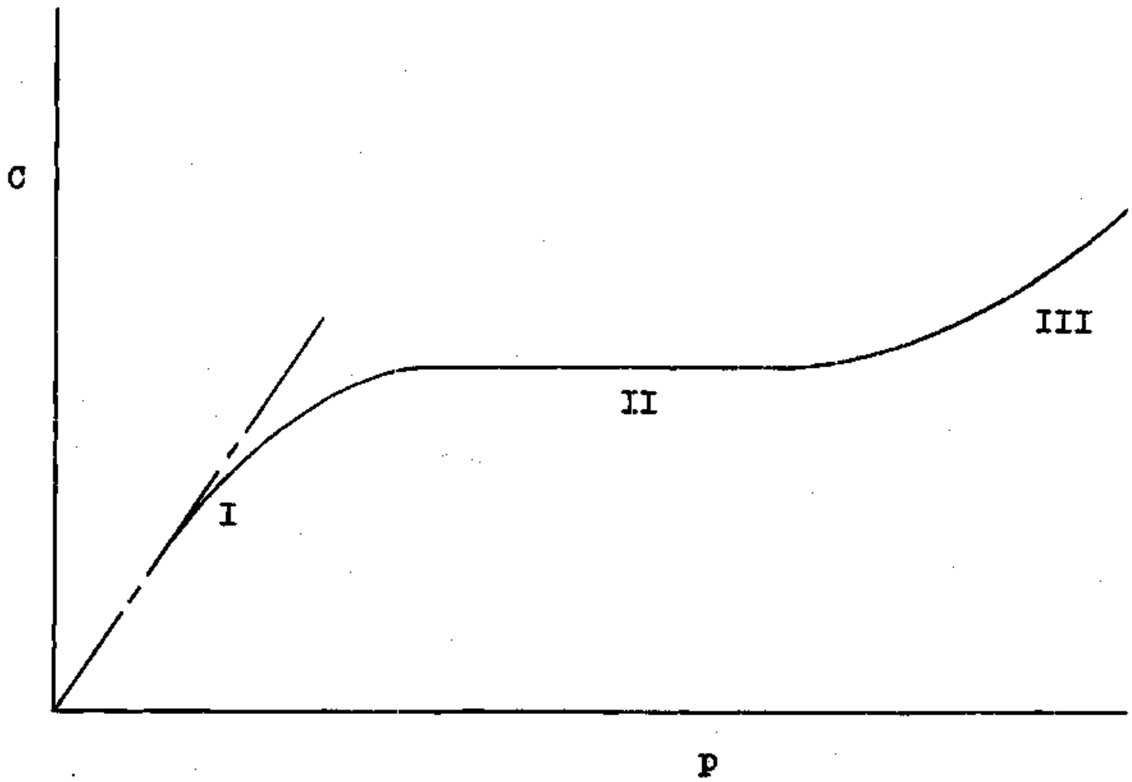


Figure 9

Variation of Coefficient C with Pressure
in a Perfect Gas

value, curve II in Fig. 9, which is proportional to the thermal conductivity of the ambient medium. For a perfect gas, the thermal conductivity is theoretically independent of pressure, down to the transition region between durves I and II in Fig. 9. As pressure is increased in the region of curve II, free convection effects eventually become significant, causing a continuous increase in C with increasing pressure (curve III).

In Appendix F, a criterion is derived by which the effect of the presence of an ambient medium can be estimated. It is assumed that the effect of conduction and convection in the ambient medium is small in comparison with the effect of radiation. Thus the temperature change in the plate brought about by conduction and convection is treated as a small perturbation to the temperature change resulting from radiation. As is shown in Appendix F, the effect of conduction and convection in the ambient medium upon the temperature history of the plate will be small if:

$$\frac{J - KP_3}{4HP_3^{\,3}} \ll 1 , \tag{132}$$

where P_3 is a mean value of $t(1,\gamma)$, the surface temperature history when conduction and convection effects are absent. The theoretically correct value of P_3, which cannot readily be determined, must lie between unity and t_e. If $P_3 = 1$ is used, or $P_3 = 3t_e/2$ if $1 \geq 3t_e/2 \geq t_e$, the criterion of equation (132) is conservative; probably a more realistic value is $P_3 = (1 + t_e)/2$. As shown in Appendix K, this assertion is borne out for at least one choice of values of the parameters G and H.

In the above paragraphs, the effect of conduction and convection in the ambient medium is considered. However, in connection with the experimental procedure discussed earlier in this chapter, an additional effect becomes important. In the test for measuring the shape factor combination $\mathcal{F}L/k$, an ambient medium was present, at essentially the same pressure as in the experimental run. In Appendix G it is shown that, under certain assumptions, the effect of conduction in the ambient medium upon the measured value of $\mathcal{F}L/k$ is such as to exactly cancel the effect upon $t(\xi, \gamma)$ during the experimental run. In arriving at this conclusion, the pressure of the ambient medium is assumed to be the same in the experimental run as in the test for measuring $\mathcal{F}L/k$. Also it is assumed that conduction and convection effects are small in comparison with the radiation effects. Thus it is concluded that conduction and convection effects in the surrounding medium have a negligible effect upon the measured temperature history $t(\xi, \gamma)$, as long as the pressure in the ambient medium is small. The exact range of pressures for which this effect is negligible was not determined.

Although conduction and convection in the ambient medium are expected to have an effect upon the measured value of diffusivity α, such an effect is not considered herein.

Other Effects.--Several additional effects, which were found to have a negligible influence upon the temperature history $t(\xi, \gamma)$ as measured in the experimental analysis, are considered in this section. The first such effect is that of the error introduced by using the mean cooling water temperature as the environmental temperature T_e, instead of using the temperature of the inner surface of the vacuum chamber walls. This choice for T_e was made because unexpected difficulties were encountered in attempting to accurately measure the temperature of the vacuum chamber walls.

The total energy per unit time transferred through the walls to the cooling water is initially given by:

$$\dot{Q} = 6.826\ v^2/R\ , \tag{133}$$

where v is the heater voltage and R its resistance. Immediately after the heaters are shut off at the start of the experimental run, this value is reduced to:

$$\dot{Q} = 3.413\ v^2/R\ , \tag{134}$$

and then continues to reduce, approaching zero asymptotically at large time.

The mass flow rate of cooling water is then given by:

$$\dot{m} = \dot{Q}/\Delta T_{H_2O} \, . \tag{135}$$

The mean temperature T_e at the inside surface of the vacuum chamber wall is given by:

$$\widetilde{T}_e = T_{H_2O} + \dot{Q}\left(\frac{\delta}{k_c A_c} + \frac{1}{C_c A_c}\right) , \tag{136}$$

in which δ is the thiciness of the wall, T_{H_2O} is the mean cooling water temperature, and the subscript "c" refers to the wall. Substituting equation (134) into equation (136) gives:

$$\widetilde{T}_e = T_{H_2O} + 3.413\,\frac{v^2}{R}\left(\frac{\delta}{k_c A_c} + \frac{1}{C_c A_c}\right) \tag{137}$$

The value of $\dot{m}$ from equation (135) is used in evaluating C_c in equation (137).

The basis for deciding whether or not $T_e - T_{H_2O}$ from equation (137) is negligible, is as follows. If the back radiation $\sigma \mathcal{F} \widetilde{T}_e^{\,4}$ is negligible in comparison with the radiation from the plate $\sigma \mathcal{F} T^4$, then replacing T_e by T_{H_2O} has a negligible effect upon the value of $t(\xi, \gamma)$.

The effect of lateral conduction in the sample was considered in designing the experiment. To estimate the effect of lateral conduction, the sample was replaced by a semi-infinite body initially at a uniform temperature. At the start of the run, this semi-infinite body commenced to lose heat from its surface at a constant rate equal to the maximum rate at which the given sample could lose heat from its lateral surface. Let L_1 represent the distance from the sample to its nearest lateral edge. Then the time at which the temperature at distance L_1 from the surface of the semi-infinite body first changed by a significant amount, was taken as an estimate of the time at which lateral conduction affects the temperature at the center of the sample.

The surface of the sample was heated by a series of heater wires located a short distance away from the surface (see Fig. 7). Under such conditions, the surface of the sample is not uniformly heated. To assess the effect of such nonuniform heating, the shape factor between the heater wires and an element of area dl on the surface parallel to the wires (see Fig. 10) was evaluated as a function of location of element dl. For this purpose, the surface was assumed to be infinite in extent, and an infinite number of equally spaced parallel wires was assumed. For the heater design used in the experimental analysis, the variation of shape factor with position of the element dl was found to be approximately one per cent, and was thus neglected.

Another effect which could cause a variation in temperature across the sample surface, is a variation of shape factor between elements on the surface and the vacuum chamber wall. To study this effect, the shape factor

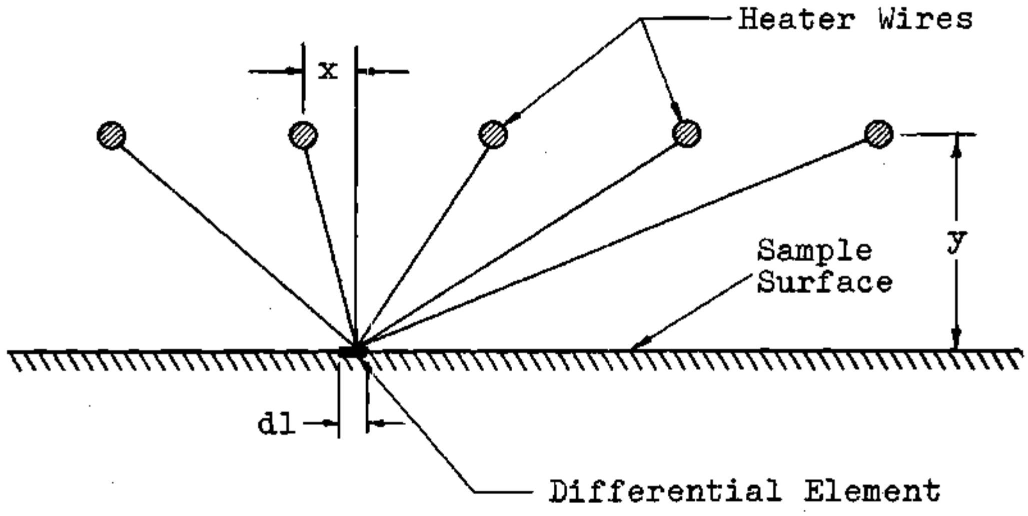


Figure 10. Shape Factor between Wires and Sample Surface

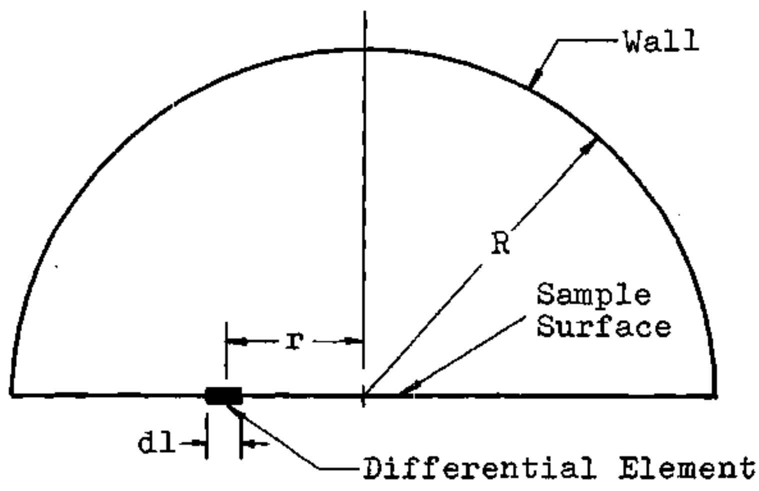


Figure 11. Shape Factor between Sample Surface and Vacuum Chamber Wall

between element dl (see Fig. 11) and the cylindrical wall was evaluated as a function of position of the element. It was found that this shape factor is independent of position of the element dl.

Certain other minor effects were considered which, however, are not discussed herein.

CHAPTER V

NUMERICAL RESULTS AND DISCUSSION

Numerical Results.--The experimental analysis described in Chapter IV was performed in the Mechanical Engineering Research Laboratory at Georgia Institute of Technology. The maximum temperature to which the sample could be raised was approximately 500°F., while the cooling water temperature varied from 75-81°F. The experimental analysis was performed in three phases. In the first phase, the shape factor combination $\mathcal{F} L/k$ was measured for several temperatures in the range 80-500°F. In the second phase, the diffusivity α was measured at several temperatures over the same range. In the third phase, seven experimental runs were performed with three different initial temperatures in the above range.

The experimental data and sample calculations from the tests for measuring shape factor and diffusivity are presented in Appendix H. The variation of shape factor with temperature is shown in Fig. 12, where the shape factors are shown for each side of the sample. The designations "red side" and "black side" refer to the color wire used to connect the respective heaters, and are retained for convenience in distinguishing between the two faces

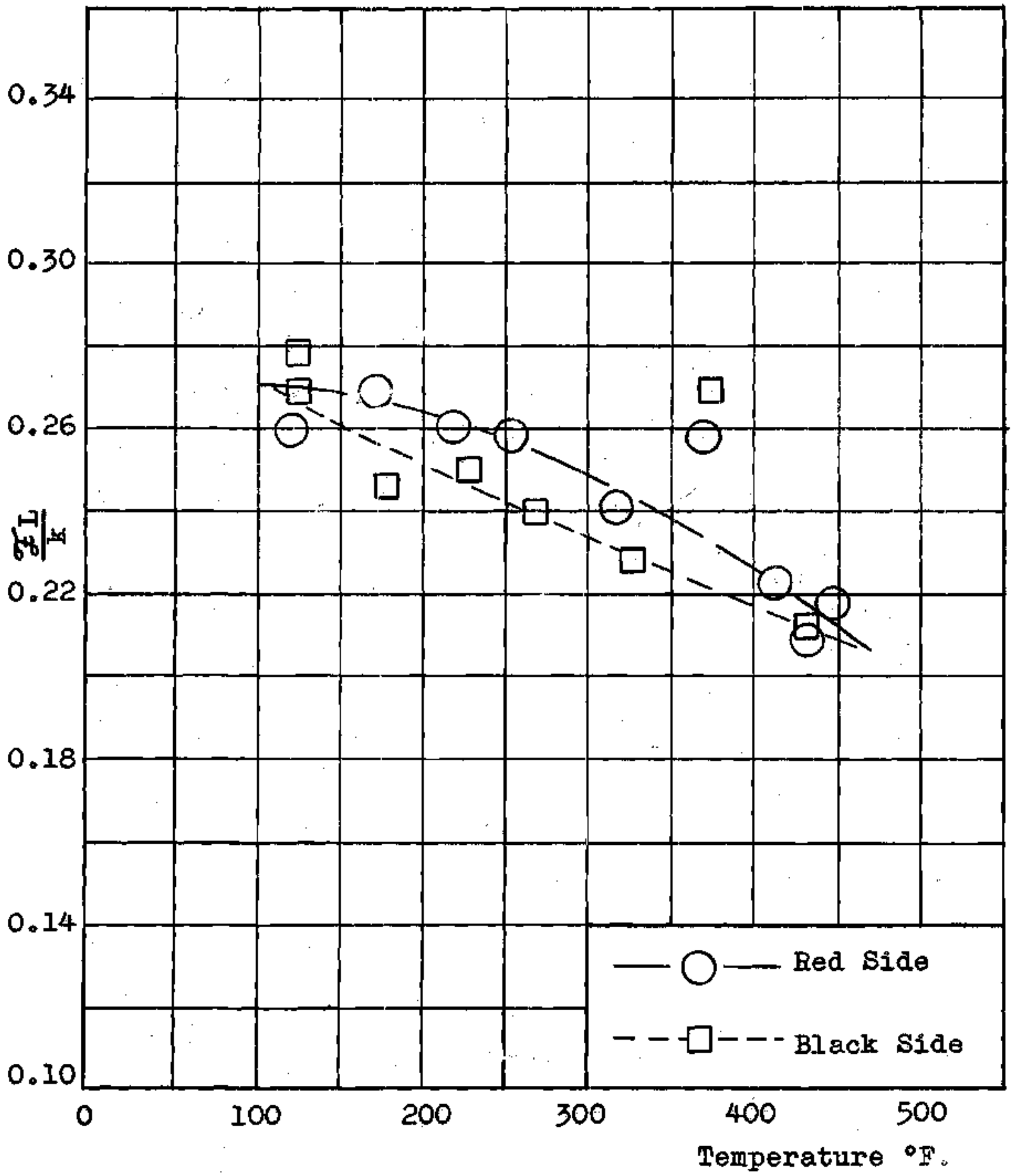


Figure 12

Variation of Shape Factor Combination with Temperature of the sample. The variation of diffusivity of the sample with temperature is shown in Fig. 13.

The third phase of the experimental analysis consisted of the experimental runs proper--the measurement of the temperature-time history of the plate in cooling down from its initial temperature to the ambient temperature. In all, seven runs were conducted during this phase of the analysis, for which certain measured and computed values are tabulated in Table 1. The column headed ΔT_{H_2O} refers to the maximum rise in temperature of the cooling water after the heaters were shut off.

Table 1. Data from Temperature-Time History

Run No.	Pressure μ Hg. abs.	T_o °F	T_e °F	ΔT_{H_2O} °F	G	H
1	5.1	504.5	81.0	0.5	0.0370	0.374
2	1.7	186.7	79.4	≈ 0.0	0.0612	0.126
3	2.8	374.5	80.1	0.2	0.044	0.254
3'	3.0	374.5	77.0	0.2	0.044	0.254
3a	60	375.0	76.3	0.2	0.044	0.254
3b	250	374.5	75.8	0.2	0.044	0.254
3c	29.25*	374.0	75.0	0.3	0.044	0.254

*in. Hg. abs.

The temperature-time histories for runs 1, 2, and 3 are plotted in Figs. 14, 15, and 16 respectively. In these figures, several experimental values are plotted, for both sides of the sample, and also for its center. The corres-

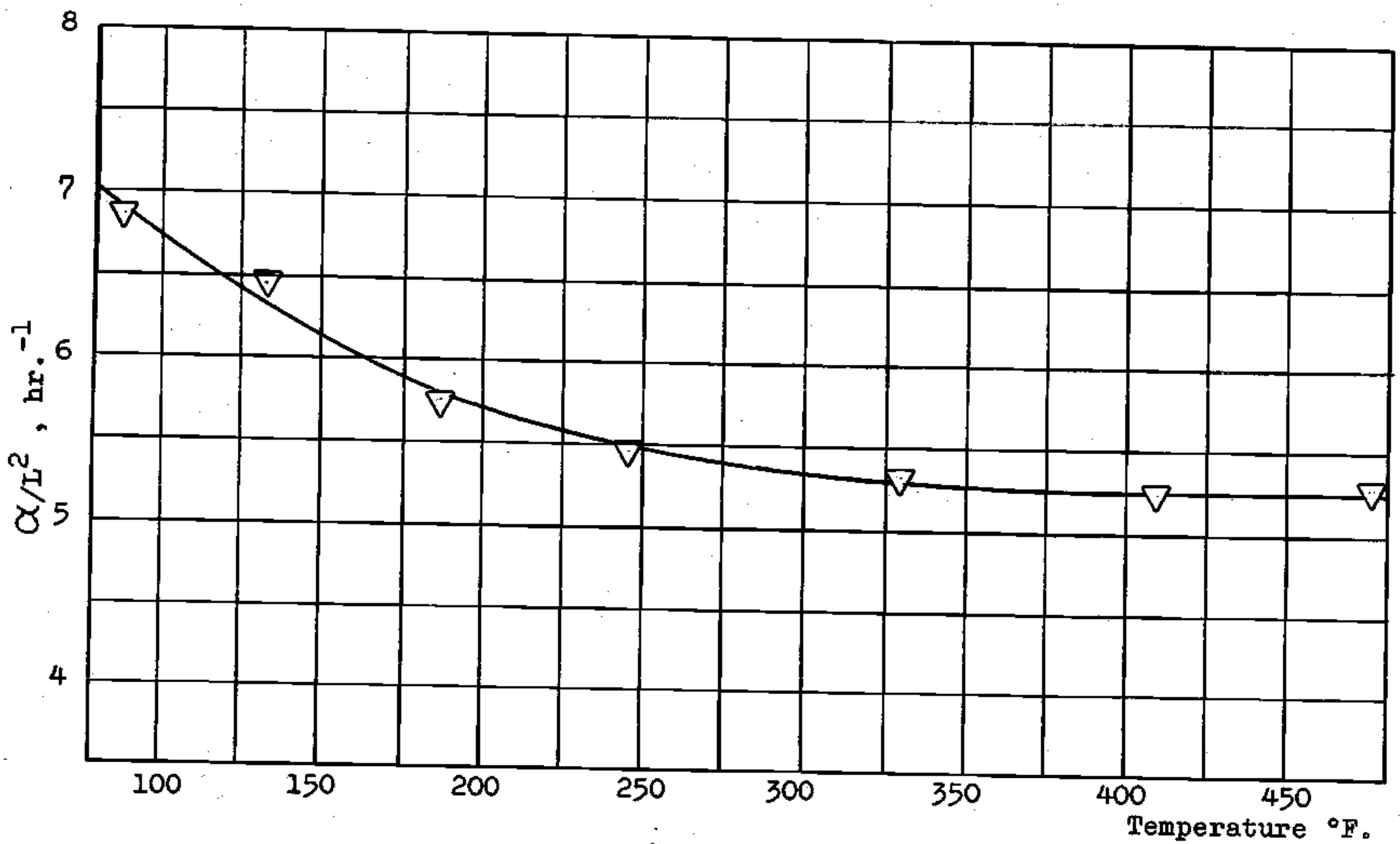


Figure 13

Variation of Diffusivity with Temperature ponding numerical solutions are also plotted for comparison. It is seen that the maximum error in the experimental values is about two per cent. In Appendix I are presented some sample calculations necessary to obtain the values plotted in Figs. 14, 15, and 16.

Figures 17, 18, and 19 show a comparison between the different methods of solutiom, with the exception of the experimental method. The numerical solution, the quadratic approximate solution, and the solution of Zerkle (28) are shown. Also shown are the short-time and long-time asymptotic solutions of Abarbanel (10).

Taking the numerical solution as correct, it is seen that the quadratic approximation is accurate to about one per cent, except for interior points in the plate for a brief period beginning at time τ_o. Zerkle's solution is accurate to about 1.5 per cent, as can be seen from the graphs. The above comments apply to the range of values of G and H represented in runs 1, 2, and 3; some comments as to the accuracy of the various methods outside this range are made in a subsequent section.

Probable Error Effects.--In this section is considered the effect of certain errors known to have been committed in performing the experiment. Although it is recognized that certain random errors are inherent in the instrumentation, for which no correction is possible, some speculations are made regarding the effect of certain systematic errors

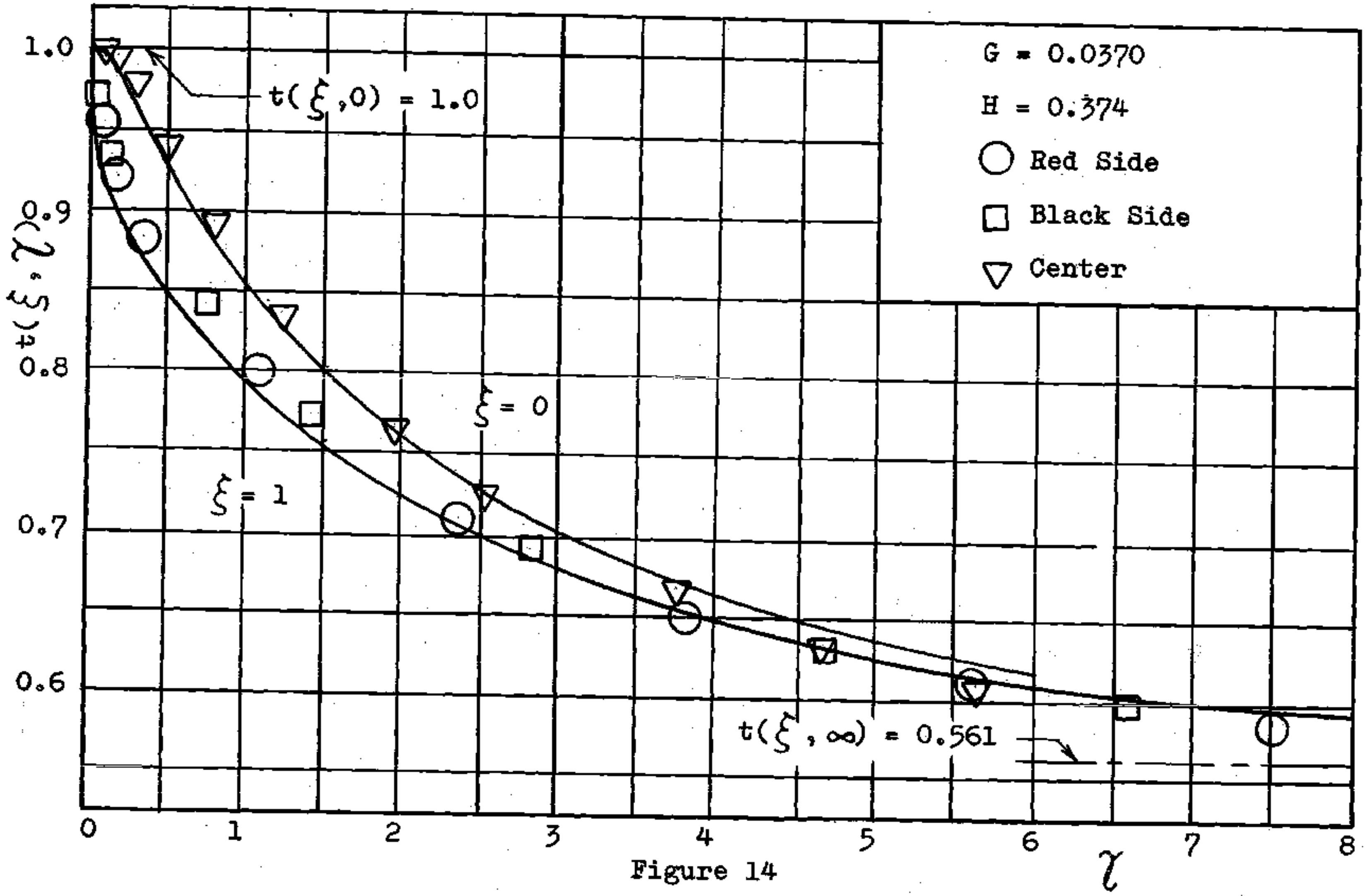


Figure 14

Experimental and Numerical Solutions for Run No. 1

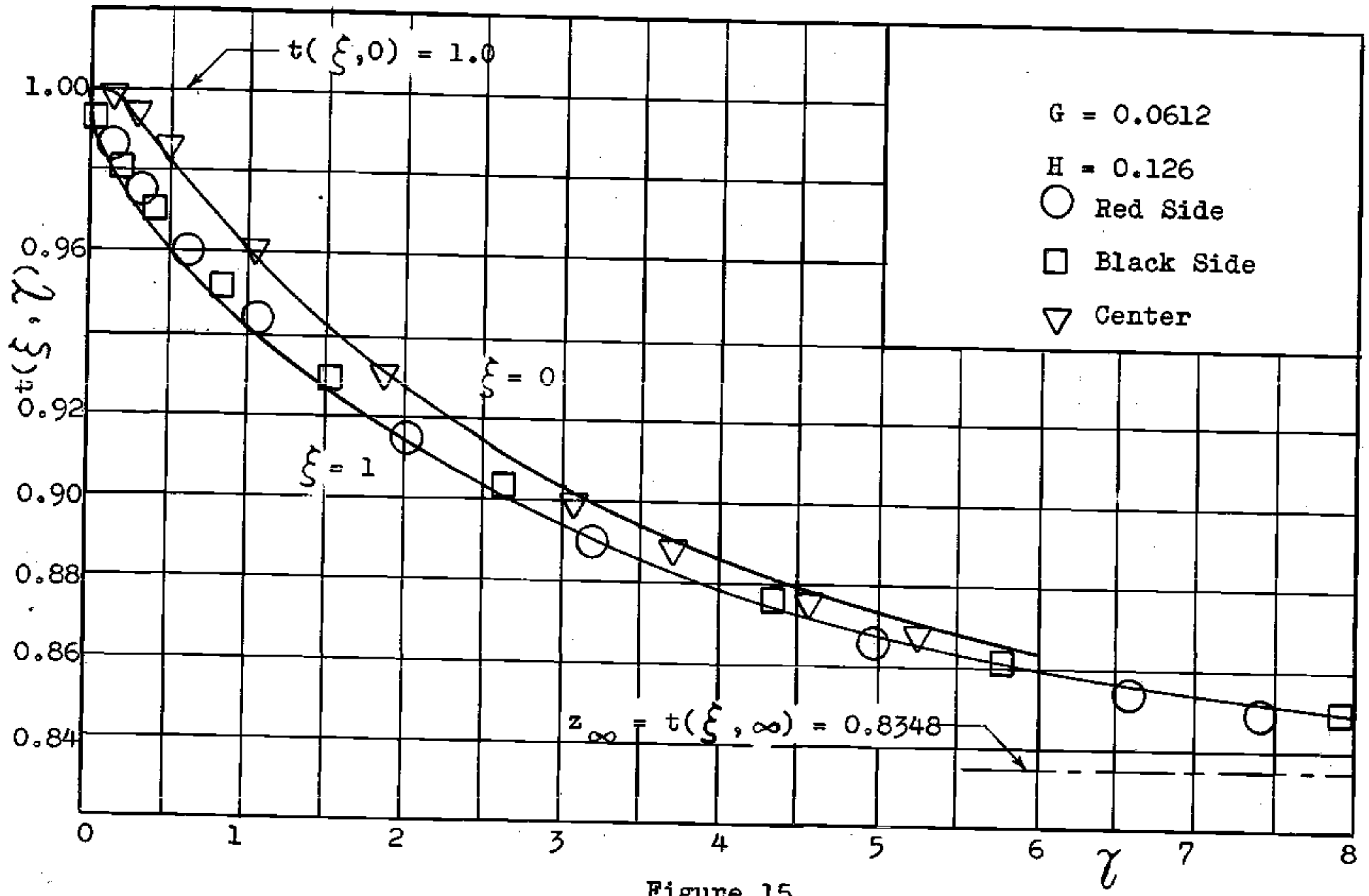


Figure 15

Experimental and Numerical Solutions for Run No. 2

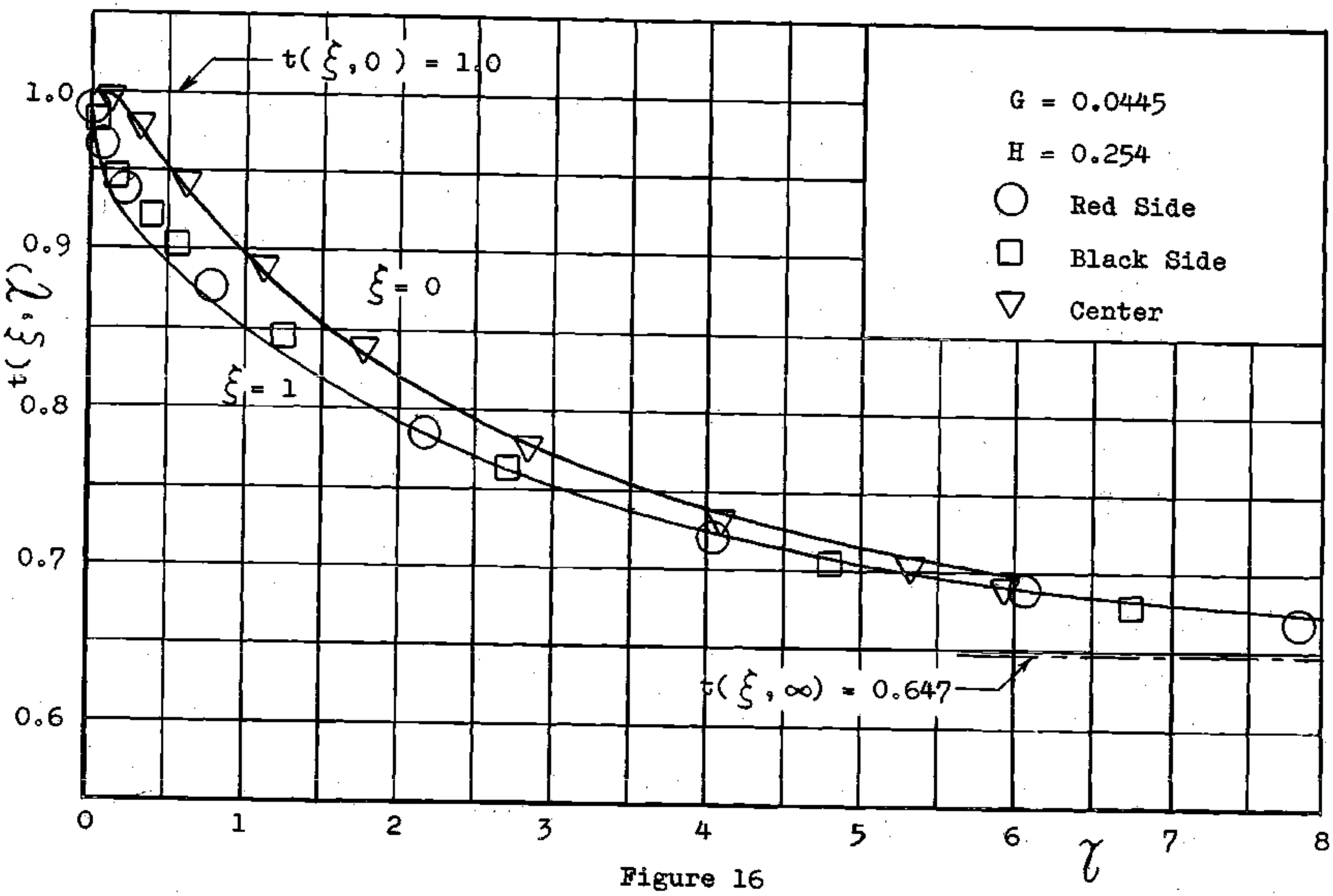


Figure 16

Experimental and Numerical Solutions for Run No. 3

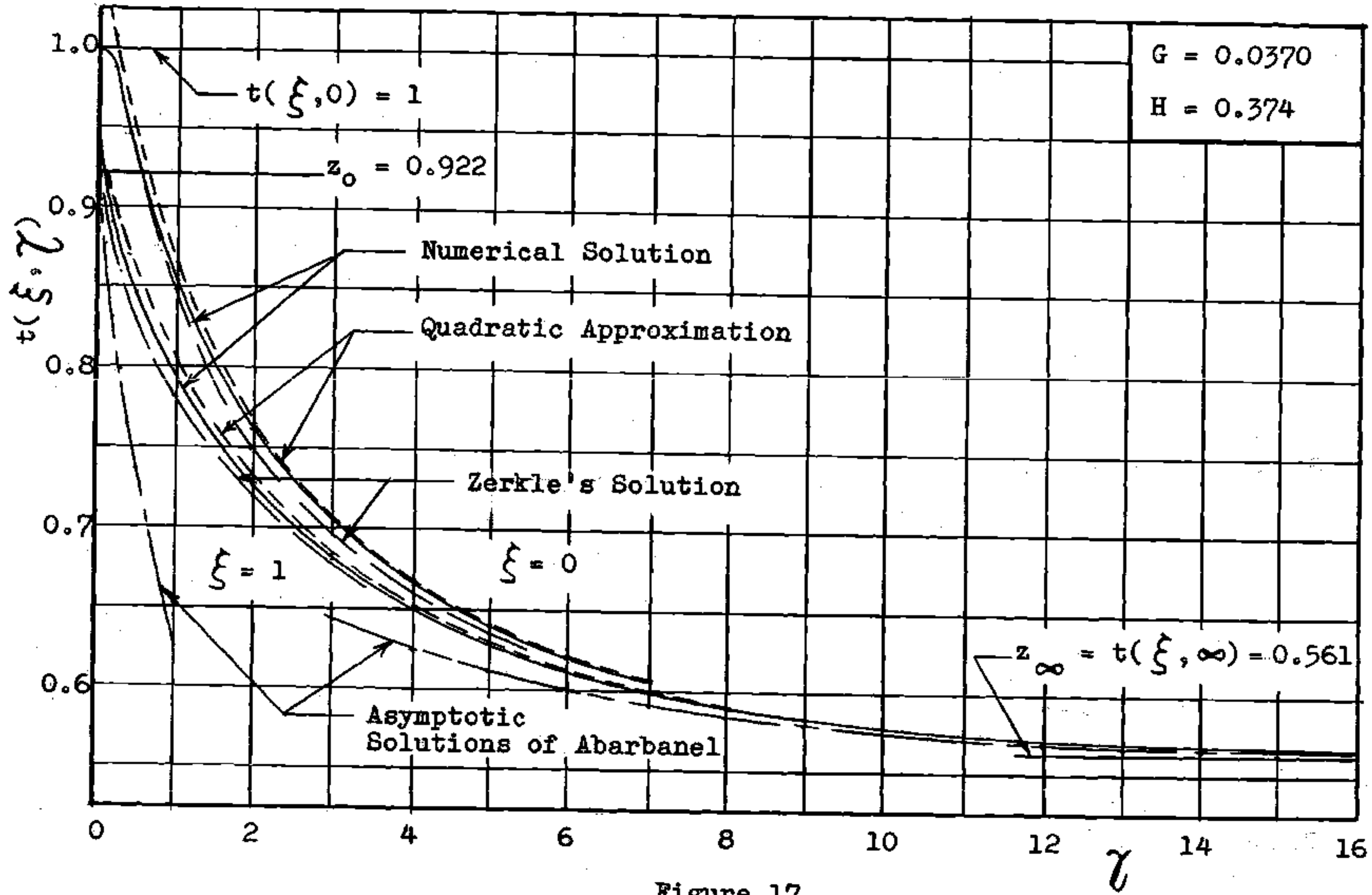


Figure 17

Comparison of Solutions for Run No. 1

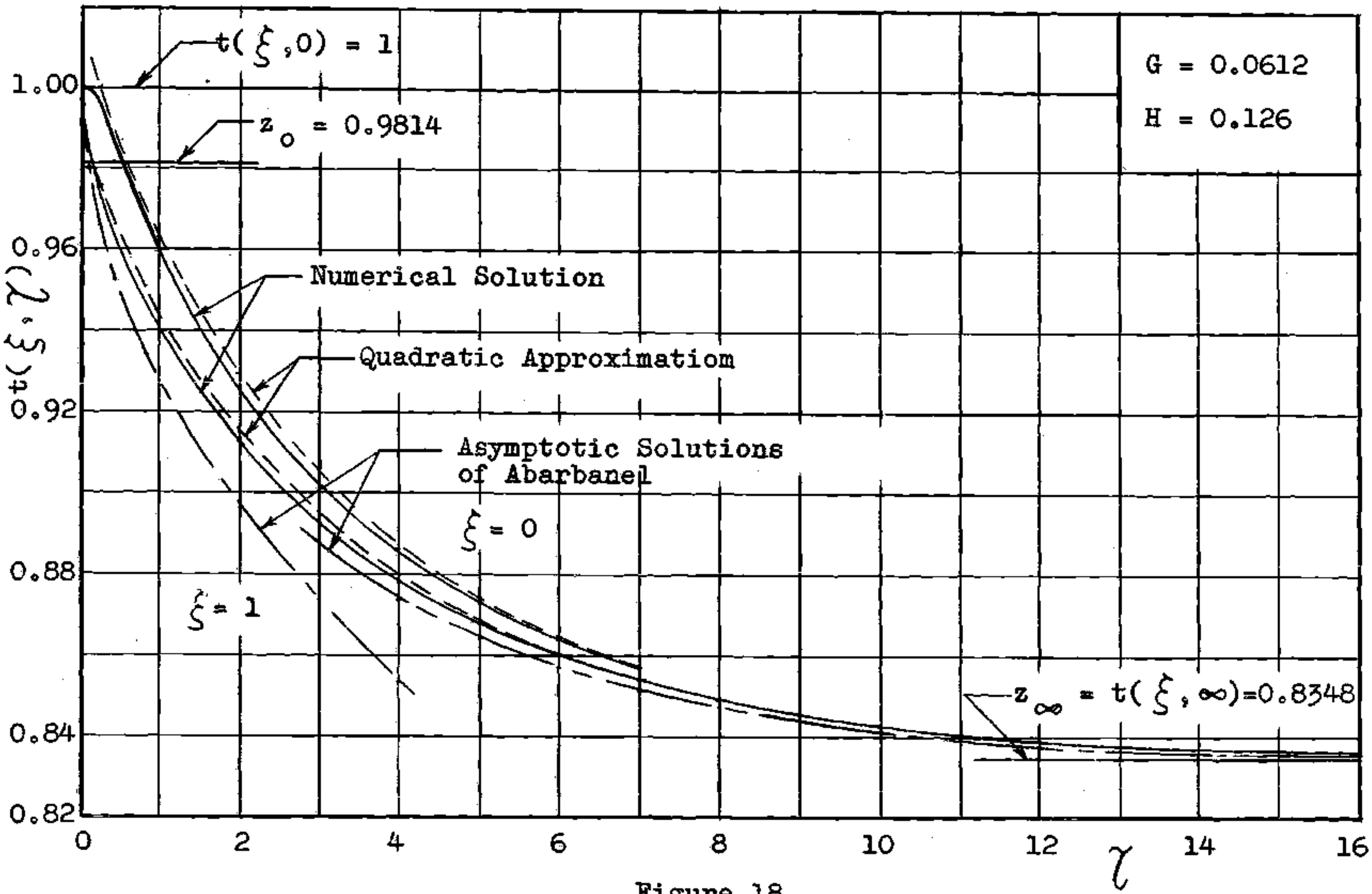


Figure 18

Comparison of Solutions for Run No. 2

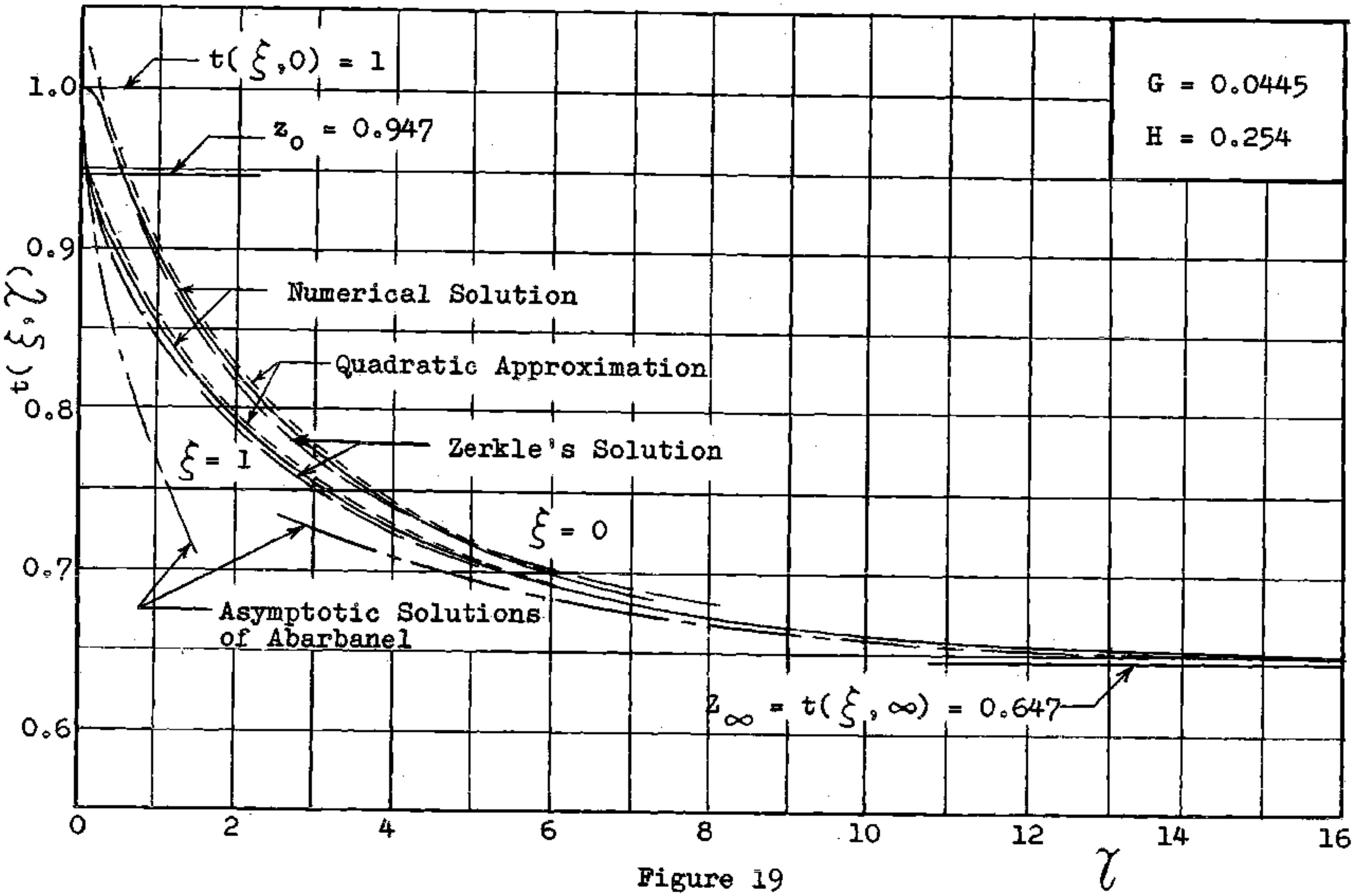


Figure 19
Comparison of Solutions for Run No. 3

known to be present.

The first effect considered is that due to the error of using the mean cooling water temperature as the environmental temperature, instead of the inside wall temperature. This effect is discussed in the section entitled "Other Effects" in Chapter IV. In Appendix J the values of the convective coefficient C_o in equation (137) between the cooling water and the vacuum chamber wall are obtained, and the inner wall surface temperatures found. The error in the back radiation is $\sigma \mathcal{F}(\widetilde{T}_e{}^4 - T_e{}^4)$, and a comparison of this quantity with the radiation $\sigma \mathcal{F} T_o{}^4$ from the sample shows the former quantity to be negligible. Hence, replacing the inner wall temperature by the mean cooling water temperature appears to be an acceptable procedure.

The thermocouples at the surfaces of the sample were deliberately imbedded slightly within the sample, to avoid the effect of a different cooling rate of the thermocouple wire than of the sample surface, due to different values of emissivity. This imbedding of the thermocouple wire--by an unknown amount--creates in itself a source of error. In Fig. 20, the experimental data for run number 3, modified as discussed below, is plotted, together with the numerical solution, with the temperature-time curve for $\xi = 0.9$ also included.

As seen in Fig. 13, the diffusivity α varies noticeably with temperature. Although the mean value of diffusivity

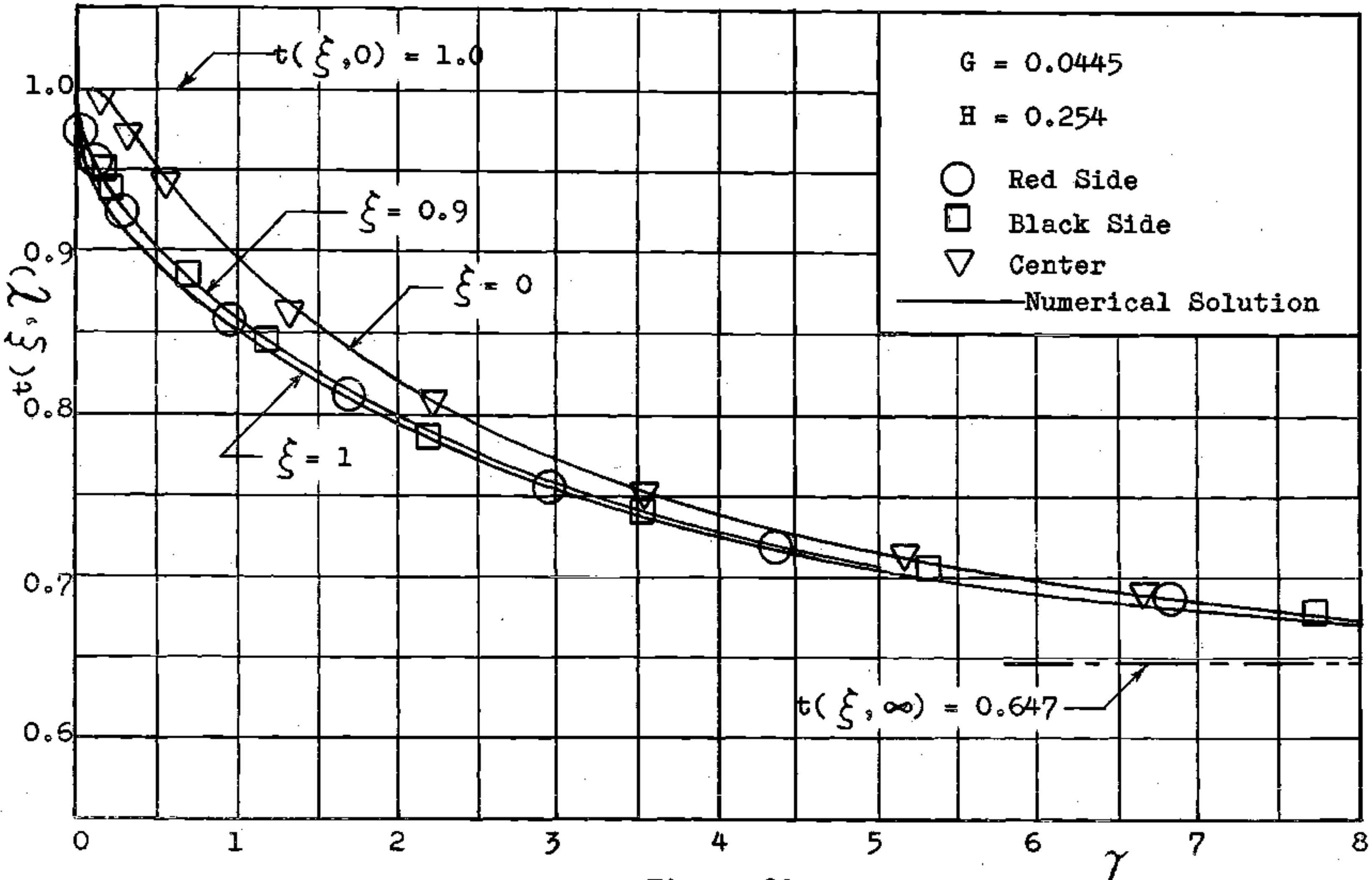


Figure 20

Experimental Data of Run No. 3 Considering Variable Diffusivity was used in reducing the experimental data for plotting Figs. 14-16, this procedure nevertheless introduces a certain error. In Fig. 20, the value of diffusivity from Fig. 13, valid at the individual temperature of each data point, was used in computing the nondimensional time γ for that data point. In this way, the variation of diffusivity with temperature was taken into account. It is not suggested that this means of accounting for variation in α is exact, or even a good approximation; it is merely interesting to observe the effect which this procedure has upon the experimental curves. It would be interesting to compare theoretically the temperature-time history based on constant α, with the history obtained from a variable α, where actual temperature is reduced to nondimensional form by using the actual value of α which holds at the particular temperature so reduced. Such a comparison is suggested as a field for future investigation.

It is seen from Fig. 20, that modifying the experimental temperatures as suggested in the preceding paragraph, effects good agreement over most of the time range, if the surface thermocouples are assumed to be imbedded by an amount L/10. It is not suggested that the thermocouples are actually imbedded this much. Instead, it is felt that other systematic error effects, whose identities are as yet unknown, account for most of the error remaining in Fig. 20.

Effect of Ambient Pressure.--In Chapter IV and Appendix G it is shown that the presence of an ambient medium at low pressure has a negligible effect upon the temperature-time history of the plate, provided that the shape factor combination $\mathcal{F}L/k$ is measured with the same ambient medium present at the same pressure. In this section, the effect of an ambient medium upon the temperature-time history, when $\mathcal{F}L/k$ has been measured in a perfect vacuum, is treated. A criterion for predicting whether or not the effects of conduction and convection in the ambient medium are negligible, is given in equation (132), and is derived in Appendix F.

As shown in Table 1, run No. 3 was duplicated but at several values of ambient pressure. Curves representing the experimental data for these runs are plotted in Fig. 21 for $\xi = 0$ and $\xi = 1$. It is seen that even with an ambient medium present at atmospheric pressure, the error in the temperature-time curve is only about six per cent. It may also be observed that the curves for pressures of 60 and 250 microns of mercury pressure are almost identical, which suggests that the coefficient C in Fig. 9 is given by curve II of that figure, a result borne out by calculation. Calculations also reveal that for the 2.3 μ Hg. pressure run, the coefficient C is about half way up on curve I in Fig. 9.

Assuming the 2.3 μ Hg. run to be correct, the

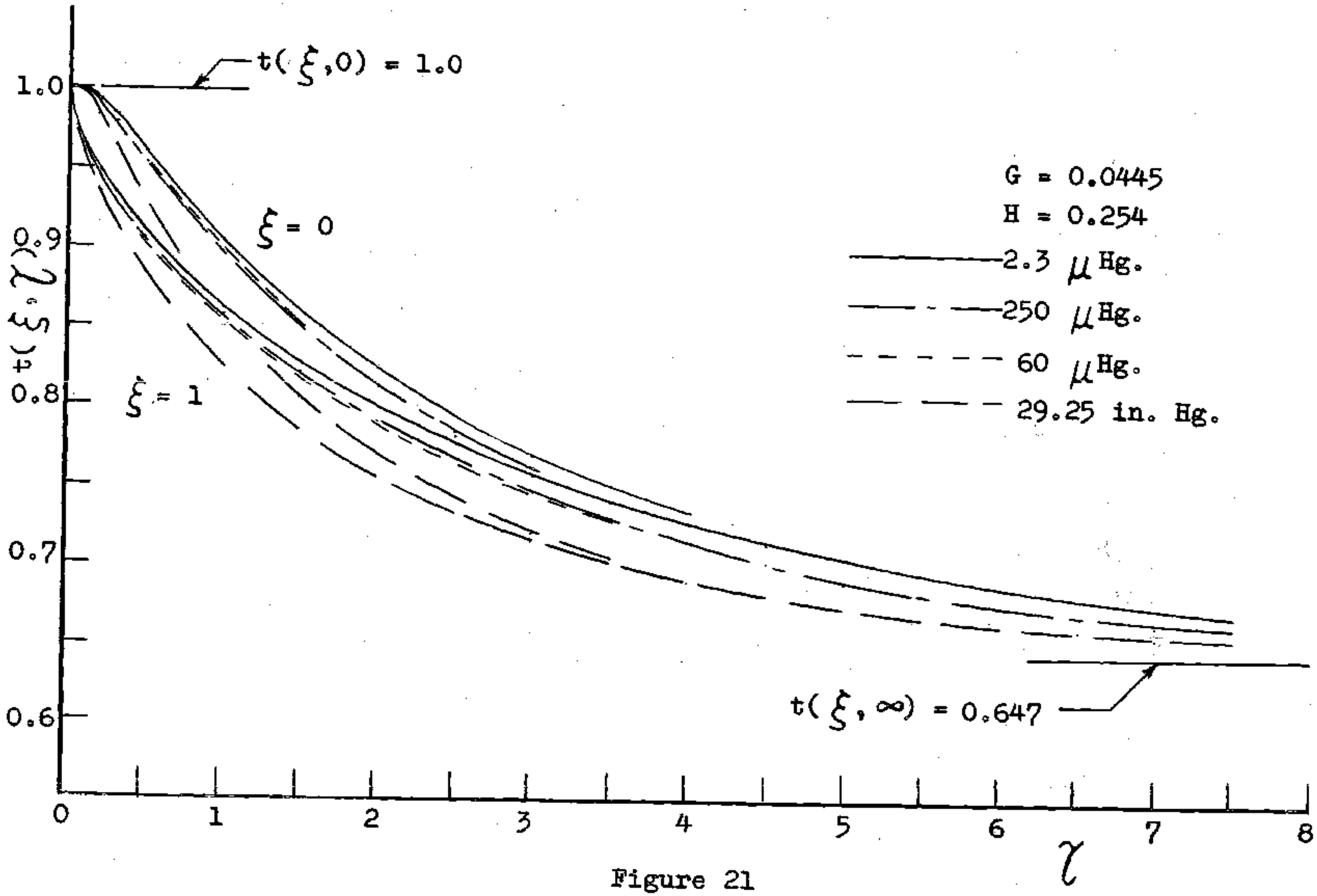
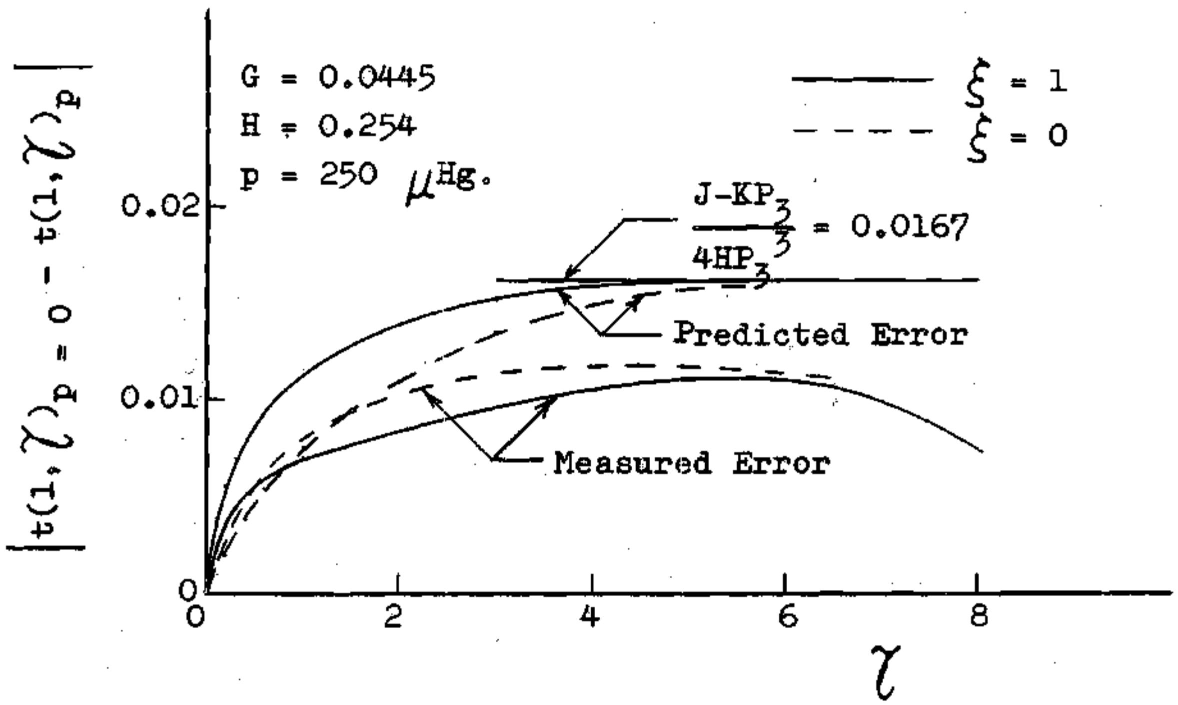


Figure 21

Experimental Results for Run No. 3 Repeated with Various Ambient Pressures absolute errors for the 250 μ Hg. and the atmospheric pressure runs are plotted as a function of time in Fig. 22. These errors are also percentage errors based on the initial temperature $t(\xi,0) = 1$. Also plotted in Fig. 22 are the predicted error curves corresponding to both pressures. These predicted error curves were obtained from the solution to the system of equations (F6)-(F8) and (F10) in Appendix F, given by Jakob (45), for the conservative case where $P_3 = 3t_e/2 = 0.97$. These predicted error curves approach the value given by equation (132) asymptotically. The values of J and K for use in equations (F10) and (132) are computed for the 250 μ Hg. run and the atmospheric pressure run in Appendix K.

From Fig. 22 it is seen that for the atmospheric pressure run, the error prediction is conservative by a factor of two. For the 250 μ Hg. run, the error prediction is conservative by a factor of one and one-half. However, for this run the measured error is only about one per cent, part of which may easily be attributed to other errors.

Extension of Range.--The range of values of G and H for which solutions were obtained in the previous sections was necessarily limited by the range of temperatures obtainable in the experimental apparatus. In order to form some idea as to the merits of the approximate and numerical solutions and Zerkle's solution, over a wider range, these solutions were obtained for a few additional values of G and H.



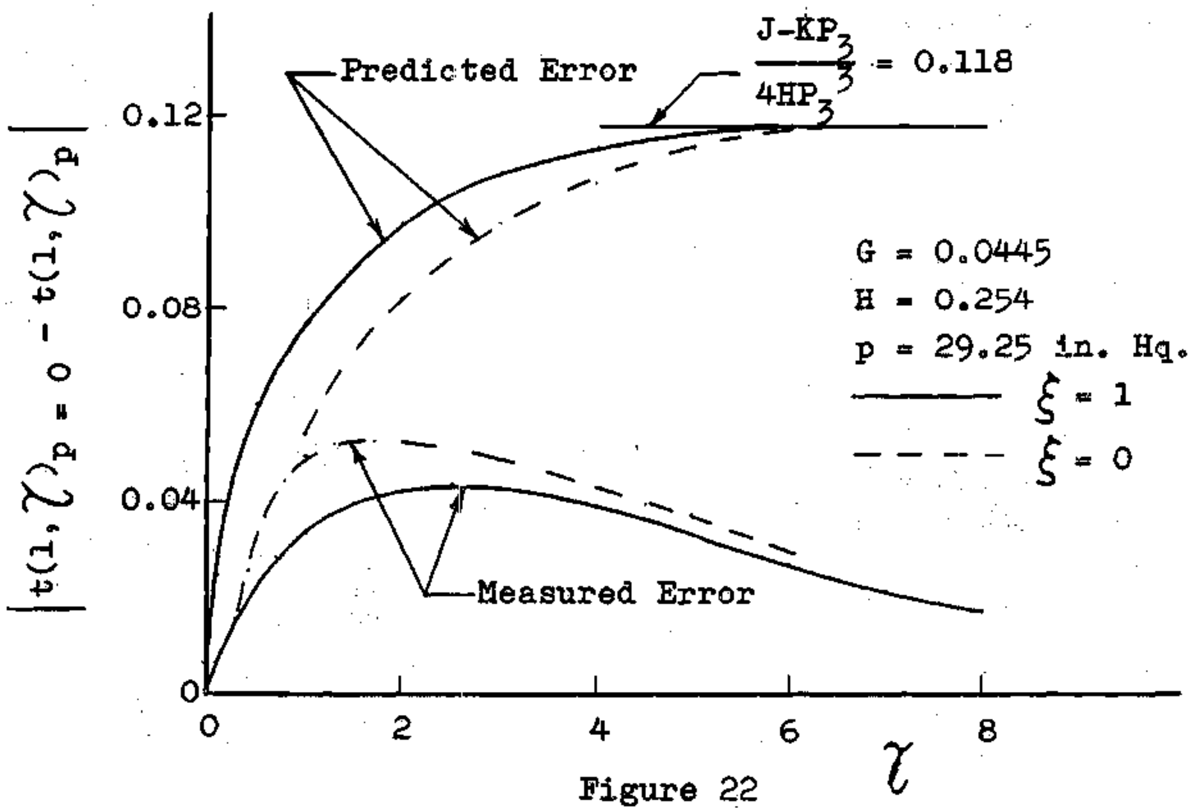


Figure 22

Measured and Predicted Errors due to Effect of Ambient Pressure

These results are tabulated in Table 2, where the maximum errors in the quadratic approximation and in Zerkle's solution are shown for $\xi = 0$ and $\xi = 1$. These errors are determined relative to the numerical solution. Two errors are given for the quadratic approximation at $\xi = 0$. The first is the maximum error, occurring at time γ_o, given by $(z_o - 1)/2$. The second is the maximum error occurring in the range $z_o \leq t(0,\gamma) \leq t_e$ for heating or $z_o \geq t(0,\gamma) \geq t_e$ for cooling.

Table 2. Errors in Quadratic Approximation and Zerkle's Solution

			$\xi = 1$		$\xi = 0$		
Run No.	G	H	Quad. Approx.	Zerkle's Sol'n	Quad. Approx.		Zerkle's Sol'n
1	0.0125	0.20	1 %	1 %	2.5 %	1 %	1.5%
2	0.040	0.00015625	0	0	0.6	0	0
3	2.0	0.00782	5	3	32	6	2
4	0.0	1.0	3	3	8.1	2.5	0.5

Although it is dangerous to generalize too freely from the limited results of Table 2, a few comments nevertheless appear to be in order. Both the quadratic approximation and Zerkle's solution give very good approximations for small values of G and H. These solutions become poorer approximations as either G or H, or both, assume larger values. On the whole, Zerkle's solution appears to be slightly closer than the quadratic approximation. However, the values of G and H in Table 2 were selected so that Zerkle's solution could be read directly from his charts without interpolation. His solution is subject to greater error where interpolation is required.

CHAPTER VI

CONCLUSIONS

The results of the analytical and experimental solutions presented herein agreed on the whole very closely with each other. The differences are of the order of magnitude of accidental errors to be expected in the instrumentation for the experimental solution.

The numerical solution presented no great difficulties, and agreed closely in every case tested with the large-time asymptotic solution, even after running for a large time interval. It is felt that the amount of computer time necessary to carry through a complete solution for a particular choice of G and H--less than thirty minutes on the Burroughs 220 computer in most cases--is not excessive. The program actually used in performing the numerical solution has a disadvantage in that the initial values must be obtained from a separate program and then fed into the computer along with the program for the numerical solution, necessitating extra handling. Also, no provision is made for automatically changing the size of the space interval as the solution proceeds--the program must be fed into the computer separately each time the interval is changed. These defects are minor and can be easily remedied if desired.

The quadratic approximation gives good results for most values of the parameters G and H of interest. The advantages of this method are that it gives an equation relating temperature and time and that this equation holds for all time values and becomes increasingly accurate at large times. Some disadvantages of this method are that the equation is not explicit in temperature, that it is rather difficult to solve, and that the surface temperature curve has an abrupt change of slope at time τ_o. Also, there may be large errors in temperature near the center of the plate for a brief period starting at time τ_o.

Zerkle's solution is also a good one, having the advantage of giving a quick answer at least as accurate as the quadratic approximation. Disadvantages of Zerkle's solution are that it usually requires interpolation not only between curves on a chart but between charts as well, and that his charts do not extend to very short or very long times.

The experimental analysis gave unexpectedly good results. It is felt that this illustrates the desirability of measuring the properties of the sample in place in the experimental apparatus, where feasible. The test for measuring diffusivity appears to give good results, whereas the test for measuring the shape factor combination $\mathfrak{F}L/k$ seems to give only fair results.

It appears there are now several methods available to obtain a good engineering approximation to the problem of transient conduction in an infinite plate with a radiation boundary condition.

CHAPTER VII

RECOMMENDATIONS

A host of new problems is suggested by the solutions presented in this paper. Of primary importance, from an academic viewpoint, is the need for an exact analytical solution to the problem treated in this paper.

Numerous examples of special cases of the generic problem involving the infinite plate suggest themselves. One problem arises upon replacing the radiation boundary condition by one of combined radiation and conduction or convection. Other problems arise when the adiabatic boundary condition is replaced by one of fixed temperature, or by one of uniform heat transfer, or by a convection boundary condition. The treatment of a composite infinite plate is suggested as an interesting problem. Also of interest is the treatment of the infinite plate with temperature-dependent properties. In all of these cases, it would be logical to attempt an approximate solution based on the heat balance integral technique, a numerical solution, and then an exact analytical solution. Some experimental work might also be devoted to the above problems.

Extension to other geometries is important, with a treatment of the infinite cylinder and sphere meriting early consideration. In these cases the heat balance integral technique, the numerical technique, and the experimental method should be applied, and finally an analytical solution obtained.

Additional work needs to be done on the problem treated herein. A reasonable quartic or quintic approximate solution based on the heat balance integral technique should be obtained. In the numerical solution a more sophisticated error estimate and stability analysis for the nonlinear boundary condition is worthy of effort. An investigation should be made to see if the numerical technique could be performed with less computer time, or to see if another technique would give equally accurate results with less computer time.

The procedure used in the experimental analysis for measuring the diffusivity of the sample seemed to give good results, and should be further investigated as to its accuracy, with a view toward using this method to measure the diffusivity of various substances. Alternatively, the solution to the problem treated in the present work forms a possible method for measuring diffusivity. The idea would be to take the diffusivity as that value which reduces the experimental temperature-time history to fit the numerical solution. This method too should be investigated as to its accuracy. The problem mentioned in the section entitled "Probable Error Effects" in Chapter V should perhaps be investigated.

APPENDIX A

DERIVATION OF DIFFERENCE EQUATION AT THE BOUNDARY $\xi = 1$

In this section is derived that portion of the numerical algorithm which applies at the boundary m = M ($\xi = 1$). The following equations apply at this boundary:

$$\frac{\partial t_{M,n}}{\partial \xi} = G - H\ t_{M,n}^4 \ ; \tag{84}$$

$$\frac{\partial^3 t_{M,n}}{\partial \xi^3} = \frac{\partial t_{M,n}}{\partial \tau} \ , \tag{A1}$$

obtained from equation (82):

$$\frac{\partial^3 t_{M,n}}{\partial \xi^3} = \frac{\partial^2 t_{M,n}}{\partial \xi\, \partial \tau} = -\ 4H\ t_{M,n}^3 \frac{\partial t_{M,n}}{\partial \tau} \ ; \tag{A2}$$

$$\frac{\partial^4 t_{M,n}}{\partial \xi^4} = \frac{\partial^2 t_{M,n}}{\partial \tau^2} \ ; \tag{A3}$$

$$\frac{\partial^5 t_{M,n}}{\partial \xi^5} = \frac{\partial^3 t_{M,n}}{\partial \xi\, \partial \tau^2}$$

$$= -\ 4H\ t_{M,n}^3 \frac{\partial^2 t_{M,n}}{\partial \tau^2} - 12H\ t_{M,n}^2 \left(\frac{\partial t_{M,n}}{\partial \tau}\right)^2 ; \tag{A4}$$

$$\frac{\partial^6 t_{M,n}}{\partial \xi^6} = \frac{\partial^3 t_{M,n}}{\partial \gamma^3} . \tag{A5}$$

In each of the above equations n = 0, 1, 2, ...

Next, the Taylor series expansion for $t_{M-1,n}$ is written:

$$t_{M-1,n} = t_{M,n} - h \frac{\partial t_{M,n}}{\partial \xi} + \frac{h^2}{2} \frac{\partial^2 t_{M,n}}{\partial \xi^2} - \frac{h^3}{6} \frac{\partial^3 t_{M,n}}{\partial \xi^3} + \frac{h^4}{24} \frac{\partial^4 t_{M,n}}{\partial \xi^4} - \frac{h^5}{120} \frac{\partial^5 t_{M,n}}{\partial \xi^5} + \frac{h^6}{720} \frac{\partial^6 t_{M-\zeta_4,n}}{\partial \xi^6} , \tag{A6}$$

valid for n = 0, 1, 2, ..., where $0 \leq \zeta_4 \leq 1$. Substituting equations (A1)-(A5) into equation (A6), and simplifying, leads to:

$$t_{M-1,n} = t_{M,n} + hH\, t_{M,n}^4 - hG + \left(\frac{h^2}{2} + \frac{2Hh^3 t_{M,n}^3}{3}\right) \frac{\partial t_{M,n}}{\partial \gamma} + \frac{Hh^5}{10} t_{M,n}^2 \left(\frac{\partial t_{M,n}}{\partial \gamma}\right)^2 + \left(\frac{h^4}{24} + \frac{Hh^5}{30} t_{M,n}^3\right) \frac{\partial^2 t_{M,n}}{\partial \gamma^2} + \frac{h^6}{720} \frac{\partial^3 t_{M-\zeta_4,n}}{\partial \gamma^3} . \tag{A7}$$

Next, the Taylor series expansions for $t_{M,n+1}$ and $t_{M,n-1}$ are written in terms of $t_{M,n}$ and the derivatives thereof. Combining these expansions gives:

$$\frac{\partial t_{M,n}}{\partial \gamma} = \frac{t_{M,n+1} - t_{M,n-1}}{2k} - \frac{k^2}{6}\,\frac{\partial^3 t_{M,n+\mu_4}}{\partial \gamma^3} \quad ; \qquad \text{(A8)}$$

$$\frac{\partial^2 t_{M,n}}{\partial \gamma^2} = \frac{t_{M,n+1} - 2t_{M,n} + t_{M,n+1}}{k^2}$$

$$- \frac{k}{6}\left(\frac{\partial^3 t_{M,n+\mu_2}}{\partial \gamma^3} - \frac{\partial^3 t_{M,n-\mu_3}}{\partial \gamma^3}\right) , \qquad \text{(A9)}$$

where $0 \leq \mu_2, \mu_3 \leq 1$ and $-1 \leq \mu_4 \leq 1$.

Substituting equations (A8) and (A9) into equation (A7), then collecting terms, finally leads to:

$$\left(3 + \frac{16}{5}\,Hh\,t_{M,n}^3 - \frac{9}{5}\,Hh\,t_{M,n}^2\,t_{M,n-1}\right) t_{M,n+1}$$

$$= t_{M-1,n} + hG + 2t_{M,n} + \frac{7}{5}\,Hh\,t_{M,n}^4 + \frac{4}{5}\,Hh\,t_{M,n}^3\,t_{M,n-1}$$

$$- \frac{9}{10}\,Hh\,t_{M,n}^2\,t_{M,n-1}^2 - \frac{9}{10}\,Hh\,t_{M,n}^2\,t_{M,n+1}^2 + P_2 , \qquad \text{(A10)}$$

valid for $n = 0, 1, 2, \ldots$, and where:

$$P_2 = \left[\frac{h^6}{432} + \frac{Hh^7 t_{M,n}^3}{324} + \frac{Hh^7 t_{M,n}^2}{360}(t_{M,n+1}\right.$$

$$\left. - t_{M,n-1})\right]\frac{\partial^6 t_{M,n+\mu_4}}{\partial \xi^6} - \frac{Hh^{13} t_{M,n}^2}{360(36)^2}\left(\frac{\partial^6 t_{M,n+\mu_4}}{\partial \xi^6}\right)^2$$

$$- \frac{h^6}{720}\frac{\partial^6 t_{M-\zeta_4,n}}{\partial \xi^6} + \left(\frac{h^6}{864} + \frac{Hh^7 t_{M,n}^3}{1080}\right)$$

$$\left(\frac{\partial^6 t_{M,n+\mu_2}}{\partial \xi^6} - \frac{\partial^6 t_{M,n-\mu_3}}{\partial \xi^6}\right). \qquad \text{(A11)}$$

in which $0 \leq \mu_2, \mu_3, \zeta_4 \leq 1$ and $-1 \leq \mu_4 \leq 1$. It is noticed that P_2 is of the order h^6.

Equation (A10) can be solved for $t_{M,n+1}$ most conveniently by an iterative process. To facilitate this process, the following terms are defined:

$$D_o = 3 + \frac{16}{5} Hh\, t_{M,n}^2 \left(t_{M,n} - \frac{9}{16} t_{M,n-1}\right); \qquad \text{(100)}$$

$$D_1 = \frac{1}{D_o}\left(t_{M-1,n} + hG + t_{M,n}\left\{2 + \frac{4}{5} Hh\, t_{M,n}\right.\right.$$

$$\left.\left.\left[-\frac{9}{8} t_{M,n-1} + t_{M,n}\left(t_{M,n-1} + \frac{7}{4} t_{M,n}\right)\right]\right\}\right); \qquad \text{(101)}$$

$$D_2 = -\frac{9}{10} hH\, t_{M,n}^2 / D_o; \qquad \text{(102)}$$

$$P_2' = P_2 / D_o. \qquad \text{(A12)}$$

Thus, equation (A10) may be rewritten in the following abbreviated form:

$$t_{M,n+1} = D_1 + D_2\, t_{M,n+1}^2 + P_2' \,, \qquad \text{(A13)}$$

for n = 0, 1, 2, ... When the truncation error term P_2' is omitted, equation (99) results.

APPENDIX B

COMPUTER PROGRAMS

Several computer programs were written in ALGOL for use on the Burroughs Algebraic Compiler at the Rich Electronic Computer Center at Georgia Institute of Technology. Three of these programs are reproduced in the following order: A program for evaluating the approximate solution; a program to provide starting values for the numerical solution; and a program for the numerical solution itself.

Program for Quadratic Approximate Solution.--This program is reproduced below. Corresponding to a particular choice of the parameters G and H, the program causes values of $t(\xi,\gamma)$ to be printed out, corresponding to values of ξ and γ which are also printed out. For each choice of G and H, an input data card is required on which the following information must be provided in order: G, H, M, N. Here M+1 is the number of values of ξ for which $t(\xi,\gamma)$ will be printed. In this program, the output format is designed on the assumption that M = 10 will be read in. The quantity N represents the number of values of γ for which $t(\xi,\gamma)$ will be printed.

Program for Starting Solution.--Jaeger's solution (2) for the semi-infinite body is used to obtain starting values for the numerical solution. A program written for this purpose is reproduced below. A set of input data on several cards must be read in. The first data card must contain values of G, H, M, and S, where M is the same as defined in connection with the approximate solution above, and where S determines the number of terms to be evaluated in the series of Jaeger's solution. Further data cards list, one entry per card, the values of γ for which values of $t(\xi, \gamma)$ will be printed out. For each such value of γ, $t(\xi, \gamma)$ will be printed for M+1 values of ξ. The last data card in the set must contain a number less than zero.

Several procedures are included in this program. These include procedures for evaluating $n!$, $C(n,m) = n!/m!(n-m)!$, the gamma function $\Gamma(x)$, and the n^{th} order integral of the complementary error function, $i^n\mathrm{erfc}(x)$.

Program for Numerical Solution.--The following input data must be read in the order listed: G, H, M, z_{end}, γ_o, $t_{0,0}$, $t_{1,0}$, $t_{2,0}$, ... , $t_{M-1,0}$, $t_{M,0}$, and $t_{M,-1}$. In the above, z_{end} is that surface temperature which terminates the solution.

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COMMENT PROGRAM FOR QUADRATIC APPROXIMATE SOLUTION TO THESIS PROBLEM $
COMMENT INSERT PROCEDURE RESULTPRINT() AT THIS POINT $
PROCEDURE RESULTPRINT(TAU,Z,DZDTAU,T(),M,P,Q,R) $
BEGIN INTEGER R      ,M,P,Q $
        EITHER IF R EQL 2 $
BEGIN   P = P + 2 $
        WRITE($$FORM1) $
        WRITE($$RESULTS,FORM2) $
        WRITE($$FORM1)                                          END $
        OTHERWISE $
BEGIN   IF P EQL 1  $  WRITE($$FORM4) $
        IF MOD (P-Q,45) EQL 1  $  WRITE($$FORM3) $
        IF MOD(P,5) EQL 1 $
BEGIN   P = P + 1 $
        WRITE($$FORM1) $
        WRITE($$RESULTS,FORM2)                                  END $
                                                                END $
OUTPUT RESULTS(TAU,Z,DZDTAU,FOR I = (1,1,M+1)$T(I)) $
FORMAT FORM1(W0)                ,
FORM2(S7.5,S8.4,S11.6,S12.4,S10.4,3S7.4,2S10.4,3S7.4,S10.4,W0),
FORM3(W1,*   TAU       Z        DZ/DTAU    X/L=  0        0.1     0.2     0.3
    .4        0.5       0.6     0.7     0.8     0.9     1.0*,W0)   ,
FORM4(W0,*   TAU       Z        DZ/DTAU    X/L=  0        0.1     0.2     0.3
    .4        0.5       0.6     0.7     0.8     0.9     1.0*,W0)  $
        RETURN  END  RESULTPRINT() $
INTEGER I,M,N,P,Q,R $
ARRAY T(100),XI(100) $
PP1..   READ($$DATA) $
        Z0 = 1.0  $  Z0A = 0.0 $
        UNTIL ABS(Z0 - Z0A) LSS(1.0**-6)Z0 $
BEGIN   Z0A = Z0 $
        EITHER IF MAX(G,H) LEQ 2.0 $
        Z0 = 1 + (G - H.Z0.Z0.Z0.Z0)/3 $
        OTHERWISE $
        Z0 = ((G + 3(1 - Z0))/H)*0.25                           END $
         EITHER IF G EQL 0.0 $
         TAU0 = (3/168H.H.Z0*8.0)(
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          Z0*8.0 + 14Z0.Z0 - 36Z0 + 21)                               $
       OR IF H EQL 0.0                                                $
       TAU0 = 0.75(Z0-1)(Z0-1)/G.G                                    $
       OTHERWISE                                                      $
       TAU0 = 1.0/24 + Z0/16G +                                       $
           (3/32H*0.5.G*1.5)LOG((H*0.5.Z0.Z0 + G*0.5)
         (H*0.5 - G*0.5)/(H*0.5.Z0.Z0 - G*0.5)(H*0.5 + G*0.5))
           (9/64H*0.25.G*1.75)(LOG((H*0.25.Z0 - G*0.25)
         (H*0.25 + G*0.25)/(H*0.25.Z0 + G*0.25)(H*0.25 -
      G*0.25)) + 2ARCTAN((H/G)*0.25) -
         2ARCTAN((H/G)*0.25.Z0))                                      $
       ZINF = (G/H)*0.25   $  ZEND = 1.0 + 0.99(ZINF - 1)             $
       DELTAZ = (ZEND - 1)/N    $  DELTAXI = 1.0/M                    $
       FOR I = (1,1,M+1)   $  XI(I) = (I-1)DELTAXI                    $
       WRITE($$PRELIM,FORM)                                           $
       P = 0     $  Q = 35                                            $
       FOR Z = (1.0,DELTAZ,Z0),(1.0,-ABS(DELTAZ),Z0)                  $
BEGIN  EITHER IF G EQL 0.0                                            $
       TAU = (3/168H.H.Z*8.0)(Z*8.0 + 14Z.Z - 36Z  + 21)              $
       OR IF H EQL 0.0                                                $
       TAU = 0.75(Z-1)(Z-1)/G.G                                       $
       OTHERWISE                                                      $
       TAU = 3(Z-1)(Z-1)/8(G-H.Z.Z.Z.Z)*2.0                           $
              3Z(Z-1)/16G(G - H.Z.Z.Z.Z)
         (3/32H*0.5.G*1.5)LOG((H*0.5.Z.Z + G*0.25)
       (H*0.25 - G*0.25)/(H*0.5.Z.Z - G*0.5)(H*0.5 + G*0.5))
         (9/64H*0.25.G*1.75)(LOG((H*0.25.Z - G*0.25)
       (H*0.25 + G*0.25)/(H*0.25.Z + G*0.25)(H*0.25 - G*0.25))
         2ARCTAN((H/G)*0.25) - 2ARCTAN((H/G)*0.25.Z))                 $
       DEL = 1 - 3(Z-1)/(G-H.Z.Z.Z.Z)                                 $
       FOR I = (1,1,M+1)                                              $
BEGIN  EITHER IF XI(I) LSS DEL                                        $
       T(I) = 1.0                                                     $
       OTHERWISE                                                      $
       T(I) = 1 + (G.XI(I) + H.Z.Z.Z.Z(1-XI(I)) + 3Z - 3 - G)*3.0/27
       (Z-1)(Z-1)                                         END         $
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        DZDTAU = 2(G-H.Z.Z.Z.Z)*3.0/3(H.Z.Z.Z(Z.Z-3Z+2) + G(Z-1))      $
        P = P+1   $  ZB = Z                                             $
        RESULTPRINT(TAU,Z,DZDTAU,T(),M,P,Q,1)                     END   $
        FOR Z = Z0                                                      $
BEGIN   TAU = TAU0                                                      $
       FOR I = (1,1,M+1)                                                $
        T(I) = 1+(G.XI(I)+H.Z0.Z0.Z0.Z0.(1-XI(I))+3Z0-3-G)*3.0/27(Z0-1)
        (Z0-1)     $  DZDTAU = 2(G-H.Z0.Z0.Z0.Z0)*3.0/3(H.Z0.Z0.Z0(Z0.Z0
         3Z0+2)+G(Z0-1))     $    P = P+1                               $
        RESULTPRINT(TAU,Z,DZDTAU,T(),M,P,Q,2)                     END   $
        FOR Z = (ZB+DELTAZ,DELTAZ,ZEND),(ZB+DELTAZ,-ABS(DELTAZ),ZEND)   $
BEGIN   EITHER IF G EQL 0.0                                             $
        TAU = TAU0 - 1.33LOG(Z/Z0) - (Z.Z.Z - Z0.Z0.Z0)/
        3H.Z0.Z0.Z0.Z.Z.Z                                               $
        OR IF H EQL 0.0                                                 $
        TAU = TAU0 + (Z - Z0)/G                                         $
        OTHERWISE                                                       $
        TAU = TAU0 + LOG((G-H.Z0.Z0.Z0.Z0)/(G-H.Z.Z.Z.Z))/3.0 +
        (1/4G*0.75.H*0.25)LOG((G*0.25+H*0.25.Z)(G*0.25-H*0.25.Z0)/
        (G*0.25-H*0.25.Z)(G*0.25+H*0.25.Z0))+(1/2G*0.75.H*0.25)(ARCTAN
        ((H/G)*0.25.Z) - ARCTAN((H/G)*0.25.Z0))                         $
        FOR I = (1,1,M+1)                                               $
        T(I) = Z - 0.5(G-H.Z.Z.Z.Z)(1-XI(I).XI(I))                      $
        DZDTAU = (G-H.Z.Z.Z.Z)/((4/3)H.Z.Z.Z+1) $  P = P+1              $
        RESULTPRINT(TAU,Z,DZDTAU,T(),M,P,Q,1)                     END   $
        GO TO PP1                                                       $
INPUT DATA(G,H,M,N)                                                     $
OUTPUT PRELIM(G,H,M,N,Z0,TAU0,ZINF,ZEND,DELTAZ,DELTAXI)                 $
FORMAT FORM(W1,*G =*,S12.8,W0,*H =*,S12.8,W0,*NUMBER OF SPACE INTERVALS
 =*,I5,W0,*NUMBER OF TIME INTERVALS =*,I5,W0,*Z ZERO =*,S12.8,W0,*TAU Z
ERO =*,S12.8,W0,*Z INFINITY =*,S12.8,W0,*Z END =*,S12.8,W0,*DELTA Z =*,
S12.8,W0,*DELTA XI =*,S12.8,W0,W0,W0,W0)                                $
        FINISH                                                          $
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SAMPLE DATA CARD--
    .0445   0.254   10   25   *   RUN NO. 3
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```
COMMENT SOLUTION OF THESIS PROBLEM BY JAEGERS METHOD - TO BE USED AS
        STARTING VALUES FOR NUMERICAL SOLUTION                          $
COMMENT INSERT PROCEDURES FACTORIAL(), C(), GAMMA(), INERFC(), AND
        PRINT() AT THIS POINT                                           $
COMMENT THE FIRST INPUT CARD CONTAINS G, H, THE NUMBER OF SPACE INTER-
        VALS M, AND THE MAXIMUM NUMBER, S, OF TERMS TO BE TAKEN IN THE
        SERIES FOR T(I).  SUBSEQUENT CARDS CONTAIN THE TIME VALUES FOR
        WHICH T(I) IS DESIRED.  THE LAST TIME CARD SHOULD HAVE A NUMBER
        LSS 0.0, THUS CAUSING THE NEXT CARD, CONTAINING NEW VALUES
        OF G, H, M, AND S, TO BE READ IN.                               $
PROCEDURE FACTORIAL(N)                                                  $
BEGIN INTEGER I, N                                                      $
        F = 1.0                                                         $
        IF N GTR 1                                                      $
BEGIN   FOR I = (2,1,N)                                                 $
        F = I.F                                                  END    $
        IF N LSS 0                                                      $
        F = 0.0                                                         $
        FACTORIAL() = F                                                 $
        RETURN END FACTORIAL()                                          $
PROCEDURE C(N,M)                                                        $
BEGIN INTEGER I, M, N                                                   $
        X = 1.0                                                         $
        EITHER IF ((N LSS M) OR (N LSS 0) OR (M LSS 0))                 $
        X = 0.99999999**49                                              $
        OR IF ((M EQL N) OR (M EQL 0) OR (N EQL 0))                     $
        X = 1.0                                                         $
        OTHERWISE                                                       $
BEGIN   FOR I = (2,1,N)                                                 $
        X = I.X                                                         $
        FOR I = (2,1,M)                                                 $
        X = X/I                                                         $
        FOR I = (2,1,N-M)                                               $
        X = X/I                                                  END    $
        C() = X                                                         $
        RETURN END C()                                                  $
```

```
PROCEDURE GAMMA(X)                                                   $     G 1
BEGIN FLOATING X,H,Y                                                 $     G 2
        H = 1.0                                                      $     G 3
        Y = X                                                        $     G 4
A1..    EITHER IF Y EQL 0.0                                          $     G 5
        H = 0.99999999**49                                           $     G 6
        OR IF Y EQL 2.0                                              $     G 7
        GO TO A2                                                     $     G 8
        OR IF Y LSS 2.0                                              $     G 9
BEGIN   H = H/Y                                                      $     G10
        Y = Y + 1.0                                                  $     G11
        GO TO A1                                              END    $     G12
        OR IF Y GEQ 3.0                                              $     G13
BEGIN   Y = Y - 1.0                                                  $     G14
        H = H.Y                                                      $     G15
        GO TO A1                                              END    $     G16
        OTHERWISE                                                    $     G17
BEGIN   Y = Y - 2.0                                                  $     G18
        H = (((((((1.6063118**-3.Y + 5.1589951**-3)Y +                     G19
        4.45114**-3)Y + 7.2110157**-2)Y + 8.211174**-2)Y +                 G20
        4.117742**-1)Y + 4.2278746**-1)Y + 0.99999998)H       END    $     G21
A2..    GAMMA() = H                                                  $     G22
        RETURN END GAMMA()                                           $     G23
PROCEDURE INERFC(N,X)                                                $     I 1
BEGIN INTEGER I, N                                                   $     I 2
        EITHER IF X EQL 0.0                                          $     I 3
        ERFCX = 1.0                                                  $     I 4
        OR IF ABS(X) GEQ 4.0                                         $     I 5
BEGIN   A1 = A2 = 1/X                                                $     I 6
        A10 = 0.99999999**49                                         $     I 7
        I = 0                                                        $     I 8
        UNTIL ((ABS(A1) LSS 1.0**-12) OR (ABS(A1) GTR ABS(A10)))     $     I 9
BEGIN   I = I+1                                                      $     I10
        A10 = A1                                                     $     I11
        A1 = -A1(2I-1)/2X.X                                          $     I12
        A2 = A2 + A1                                          END    $     I13
```

```
         ERFCX = EXP(-X.X)A2/1.7724538                        END   $    I14
         OTHERWISE                                                  $    I14A
BEGIN    EITHER IF X LSS -2.0                                       $    I15
BEGIN    A = -3.0                                                   $    I16
         C0 = 1.9999778                                             $    I17
         GO TO PP1                                            END   $    I18
         OR IF X LSS 0.0                                            $    I19
BEGIN    A = -1.0                                                   $    I20
         C0 = 1.8427008                                             $    I21
         GO TO PP1                                            END   $    I22
         OR IF X LSS 2.0                                            $    I23
BEGIN    A = 1.0                                                    $    I24
         C0 = 0.15729921                                            $    I25
         GO TO PP1                                            END   $    I26
         OTHERWISE                                                  $    I27
BEGIN    A = 3.0                                                    $    I28
         C0 = 2.20904**-5                                           $    I29
         GO TO PP1                                            END   $    I30
PP1..    Y = X-A                                                    $    I31
         C1 = -2EXP(-A.A)/1.7724538                                 $    I32
         D = C1.Y                                                   $    I33
         ERFCX = C0 + D                                             $    I34
         I = 1                                                      $    I35
         UNTIL ABS(D) LSS 1.0**-10                                  $    I36
BEGIN    I = I+1                                                    $    I37
         C2 = -2A.C1 - 2(I-2)C0                                     $    I38
         C0 = C1                                                    $    I39
         C1 = C2                                                    $    I40
         D = D(C1/C0)Y/I                                            $    I41
         ERFCX = ERFCX + D                                END   END   $    I42
         IERFCX = EXP(-X.X)/1.7724538 - X.ERFCX                     $    I43
         EITHER IF N LSS 0                                          $    I44
         ANS = 0.99999999**49                                       $    I45
         OR IF N EQL 0                                              $    I46
         ANS = ERFCX                                                $    I47
         OR IF N EQL 1                                              $    I48
```

```
          ANS = IERFCX                                              $
          OTHERWISE                                                 $
 BEGIN    P1 = IERFCX                                               $
          P2 = ERFCX                                                $
          FOR I = (2,1,N)                                           $
 BEGIN    ANS = (P2 - 2X.P1)/2I                                     $
          P2 = P1                                                   $
          P1 = ANS                                                  $
          INERFC() = ANS                              END   END     $
          RETURN END INERFC()                                       $
 PROCEDURE PRINT(G,H,M,TAU,DZDTAU,T(),P,R$Q)                        $
 BEGIN    INTEGER I, J, P, R, M, Q                                  $
          IF (MOD(P,55) GEQ 50) OR (P EQL 0)                        $
 BEGIN    WRITE($$RESULTS1,FORM1)                                   $
          Q = 0                                                     $
          WRITE($$RESULTS2,FORM2)                           END     $
          FOR J = (6,4,M)                                           $
          WRITE($$RESULTS3,FORM3)                                   $
          WRITE($$RESULTS4,FORM4)                                   $
          Q = P + 3 + ((M-2)/4)                                     $
 OUTPUT RESULTS1(G,H,M),                                            $
        RESULTS2(TAU,DZDTAU,FOR I=(1,1,MIN(M,5))$T(I)),
        RESULTS3(FOR I=(J,1,MIN(J+3,M))$T(I)),
        RESULTS4(T(M+1),R)
 FORMAT FORM1(W1,*G =*,S15.8,B10,*H =*,S15.8,B10,*NO. OF SPACE INTERVALS $
  =*,I5,W0,W0,*       TAU          DZDTAU
  (0,DELTA(X/L),1)                                       T FOR X/L =
                                          Z             NO.*,W0,W0)
 ,      FORM2(2S12.8,2S17.8,3S12.8,W0),
        FORM3(B46,4S12.8,W0),
        FORM4(B94,S17.8,I9,W0,W0)                                   $
         RETURN END PRINT()                                         $
         INTEGER I, J, M, N, P, Q, R, S                             $
         ARRAY XI(50), T(50), A(100), B(100)                        $
 PP1..   READ($$CONSTANTS)                                          $
         FOR I = (1,1,M+1)                                          $
         XI(I) = (FLOAT(I-1))/M                                     $
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```
         A(1) = 2(G-H)/1.7724538                                    $
         B(1) = 4A(1)                                               $
         FOR I = (2,1,S)                                            $
BEGIN    A(I) =-I.H.GAMMA(0.5(I+1))B(I-1)/GAMMA(0.5(I+2))           $
         B(I) = 4A(I)                                               $
         FOR J = (1,1,I-1)                                          $
         B(I) = B(I) + (4C(I-1,I-J) - C(I-1,J))A(J)B(I-J)      END  $
         P = 0                                                      $
PP2..    READ($$TIME)                                               $
         IF TAU LSS 0.0                                             $
         GO TO PP1                                                  $
         Z = 1.0                                                    $
         U = V = 1.0                                                $
         I = 0                                                      $
         DZDTAU = 0.0                                               $
         UNTIL (ABS(U) + ABS(V)) LSS 1.0**-8                        $
BEGIN    I = I+1                                                    $
         U = A(I).TAU*(I/2.0)/FACTORIAL(I)                          $
         Z = Z + U                                                  $
         V = A(I).TAU*((I-2.0)/2.0)/2FACTORIAL(I-1)                 $
         DZDTAU = DZDTAU + V                                   END  $
         FOR I = (M,-1,1)                                           $
BEGIN    U = 1.0                                                    $
         T(I) = 1.0                                                 $
         R = 0                                                      $
         UNTIL (ABS(U) LSS 1.0**-8) OR (R EQL S)                    $
BEGIN    R = R+1                                                    $
         U = (4TAU)*(R/2.0)GAMMA(0.5(R+2.0))A(R)
         INERFC(R,(1-XI(I))/2SQRT(TAU))/FACTORIAL(R)                $
         T(I) = T(I) + U                                  END  END  $
         T(M+1) = Z                                                 $
         PRINT(G,H,M,TAU,DZDTAU,T(),P,R$Q)                          $
         P = Q                                                      $
         GO TO PP2                                                  $
INPUT CONSTANTS(G,H,M,S),
      TIME(TAU)                                                     $
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```
COMMENT NO. IN THE PRINT OUT REFERS TO THE NUMBER OF TERMS EVALUATED
      IN THE SERIES FOR T AT X/L = 0                                     $
      FINISH                                                             $

SAMPLE DATA CARDS--
    .0445    0.254    10    10    *    RUN NO. 3
    .07333333                     *    RUN NO. 3
    .075                          *    RUN NO. 3
   -1.0                           *    RUN NO. 3
```

```
COMMENT NUMERICAL SOLUTION TO THESIS PROBLEM                          $
COMMENT INSERT PROCEDURE PRINT AT THIS POINT                          $
PROCEDURE PRINT(G,H,M,TAU,DZDTAU,T(),T1,P,D$Q)                        $
BEGIN    INTEGER I, J, P, R, M, Q                                     $
         R = P                                                        $
         IF (MOD(P,55) GEQ 50) OR (P EQL 0)                           $
BEGIN    WRITE($$RESULTS1,FORM1)                                      $
         R = 0                                              END       $
         WRITE($$RESULTS2,FORM2)                                      $
         FOR J = (6,4,M)                                              $
         WRITE($$RESULTS3,FORM3)                                      $
         WRITE($$RESULTS4,FORM4)                                      $
         Q = R + 3 + ((M-2)/4)                                        $
OUTPUT RESULTS1(G,H,M),
       RESULTS2(TAU,DZDTAU,FOR I=(1,1,MIN(M,5))$T(I),T1),
       RESULTS3(FOR I=(J,1,MIN(J+3,M))$T(I)),
       RESULTS4(T(M+1),D)                                             $
FORMAT FORM1(W3,*G =*,S15.8,B10,*H =*,S15.8,B10,*NO. OF SPACE INTERVALS
 =*,I5,W0,W0,*       TAU          DZDTAU                T FOR X/L =
(0,DELTA(X/L),1)                        Z                DIFF.*,W0,W0)
     , FORM2(2S12.8,2S15.8,3S12.8,S15.8,W0),
       FORM3(B42,4S12.8,W0),
        FORM4(B90,2S15.8,W0,W0)                                       $
        RETURN END PRINT()                                            $
INTEGER I, J, M, N, N1, P, R, Q, L                                    $
ARRAY T(16,140)                                                       $
PP1..   READ($$CONSTANTS)                                             $
        DELTATAU = 1.0/6M.M                                           $
        N = 6M.M/10 + 2     $  N1 = N - 2                             $
        HS = 1.0/M                                                    $
        AA = 3.2HS.H                                                  $
        AB = HS.G                                                     $
        AC = 0.8H.HS                                                  $
        AD = -0.9HS.H                                                 $
        TAU = TAU0                                                    $
        P = R = 0             $ T(M+1,N) = 1.0                        $
```

```
        PRINT(G,H,M,TAU,(T(M+1,2) - T(M+1,1))/DELTATAU,T(,2),T(M+1,1),         15A
        P,T(M+1,2) - T(M+1,1)$Q)                                        $     15B
        P=Q                                                             $     15C
        UNTIL ((ZEND GTR 1.0) AND (T(M+1,N) GEQ ZEND))                         16
        OR ((ZEND LSS 1.0) AND (T(M+1,N) LEQ ZEND))                     $     17
BEGIN   FOR J = (3,1,N)                                                 $     18
BEGIN   FOR I = (2,1,M)                                                 $     19
        T(I,J) = (T(I+1,J-1) + 4T(I,J-1) + T(I-1,J-1))/6                $     20
        T(1,J) = (T(2,J-1) + 2T(1,J-1))/3                               $     21
        E0 = 3 + AA. T(M+1,J-1)T(M+1,J-1)(T(M+1,J-1) -                         22
               .5625T(M+1,J-2))                                         $     23
        E1 = (T(M,J-1) + AB + T(M+1,J-1)(2 + AC.T(M+1,J-1)                     24
               (-1.125T(M+1,J-2)T(M+1,J-2) + T(M+1,J-1)                        25
               (T(M+1,J-2) + 1.75T(M+1,J-1))))))/E0                     $     26
        E2 = AD.T(M+1,J-1)T(M+1,J-1)/E0                                 $     27
        T(M+1,J) = T(M,J) + T(M+1,J-1) - T(M,J-1)                       $     28
        E3 = 1.0                                                        $     29
        E30 = 0.0                                                       $     30
        UNTIL ABS(E3 - E30) LSS 5.0**-8                                 $     31
BEGIN   E30 = E3                                                        $     32
        E3 = E2.T(M+1,J)T(M+1,J)                                        $     33
        T(M+1,J) = E1 + E3                                END   END     $     34
        TAU = TAU + DELTATAU.N1                                         $     35
        DZDTAU = (T(M+1,N) - T(M+1,N-1))/DELTATAU                       $     36
        PRINT(G,H,M,TAU,DZDTAU,T(,N),T(M+1,N-1),P,                            38
        T(M+1,N) - T(M+1,N-1)$Q)                                        $     38A
        P = Q                                                           $     39
        T(M+1,1) = T(M+1,N-1)                                           $     40
        FOR I = (1,1,M+1)                                               $     41
        T(I,2) = T(I,N)                                         END     $     42
        GO TO PP1                                                       $     43
INPUT  CONSTANTS(G,H,M,ZEND,TAU0,FOR I=(1,1,M+1)$T(I,2),T(M+1,1))       $     44
        FINISH                                                          $     46
```

```
SAMPLE DATA CARDS--
     .0445      0.254     10     0.68     0.075                                          *   RUN NO. 3
     .99977379    0.99950490    0.99897630    0.99799829                                 *   RUN NO. 3
     .99629353    0.99349175    0.98914491    0.98277051                                 *   RUN NO. 3
     .97392120    0.96226842    0.94768148    0.94815847                                 *   RUN NO. 3
```

APPENDIX C

FORMAL DEMONSTRATION OF STABILITY

The stability of equations (97) and (98) is to be demonstrated in this section. The presentation is analogous to that given by Hildebrand (43).

From equations (97) and (98), making use of equation (95):

$$t_{m,n+1} = \frac{t_{m+1,n} + 4t_{m,n} + t_{m-1,n}}{6} , \qquad \text{(C1)}$$

$$m = -1,\ 0,\ 1,\ 2,\ \ldots ,\ M-1, \quad n = 0,\ 1,\ 2,\ \ldots .$$

The exact temperature, designated by $\bar{t}_{m,n}$, must satisfy the following difference equation:

$$\bar{t}_{m,n+1} = \frac{\bar{t}_{m+n} + 4\bar{t}_{m,n} + \bar{t}_{m-1,n}}{6} + P_1 , \qquad \text{(C2)}$$

$$m = -1,\ 0,\ 1,\ 2,\ \ldots ,\ M-1 , \quad n = 0,\ 1,\ 2,\ \ldots .$$

The total error is defined by:

$$e_{m,n} = \bar{t}_{m,n} - t_{m,n} , \qquad \text{(C3)}$$

$$m = -1,\ 0,\ 1,\ 2,\ \ldots ,\ M-1, \quad n = 0,\ 1,\ 2,\ \ldots .$$

Combining equations (C1)-(C3) gives:

$$e_{m,n+1} = \frac{e_{m+1,n} + 4e_{m,n} + e_{m-1,n}}{6} + P_1 \, . \qquad \text{(C4)}$$

To establish stability, it will be necessary to examine the complementary solution to equation (C4).

The complementary solution to equation (C4) is the solution to the related homogeneous equation:

$$e_{m,n+1} = \frac{e_{m+1,n} + 4e_{m,n} + e_{m-1,n}}{6} \, , \qquad \text{(C5)}$$

$$n = 0, 1, 2, \ldots ,$$

subject to the boundary conditions:

$$\frac{\partial e_{o,n}}{\partial \xi} = 0 \quad \text{or} \quad e_{-1,n} = e_{1,n} \, , \quad n = 0, 1, 2, \ldots , \text{(C6)}$$

$$\frac{\partial e_{M,n}}{\partial \xi} = 0 \quad \text{or} \quad e_{M+1,n}=e_{m-1,n} , \quad n = 0, 1, 2, \ldots , \text{(C7)}$$

and to an arbitrary initial condition. The boundary condition of equation (C7) assumes that the temperature on the boundary $\xi = 1$ is determined exactly by the numerical algorithm of equations (97)-(102). Although this is obviously not true, it is a proper assumption for the purpose of establishing the stability of equations (97) and (98). In equations (C5)-(C7), $m = -1, 0, 1, 2, \ldots , M, M+1$.

To solve equations (C5)-(C7), the method of separation of variables is used. In this method, it is assumed that $e_{m,n}$ may be represented by:

$$e_{m,n} = M_m N_n, \tag{C8}$$

where M varies only with m and N varies only with n. Substituting equation (C8) into equation (C5) then gives:

$$\frac{N_{n+1}}{N_n} = \frac{M_{m+1} + 4M_m + M_{m-1}}{6M_m} = \lambda \ , \tag{C9}$$

where λ is a constant to be evaluated later. From equation (C9), then, two ordinary difference equations are obtained:

$$N_{n+1} - \lambda N_n = 0 \ , \tag{C10}$$

$$M_{m+1} + (4 - 6\lambda)M_m + M_{m-1} = 0 \ . \tag{C11}$$

The solution to equation (C10) is:

$$N_n = \lambda^n \ , \tag{C12}$$

while that solution to equation (C11) which also satisfies equation (C6) is:

$$M_m = C \cos \phi m \ , \tag{C13}$$

where:

$$\cos \phi = -(2-3\lambda) \ . \tag{C14}$$

Thus a solution to equation (C5) is:

$$e_{m,n} = C\ \lambda^{n} \cos \phi m\ . \tag{C15}$$

Substituting equation (C15) into equation (C7) gives:

$$\cos \phi(M+1) = \cos \phi(M-1)\ ,$$

from which:

$$\sin \phi M \sin \phi = 0\ . \tag{C16}$$

Since $\sin \phi \neq 0$, then $\sin \phi M = 0$, and:

$$\phi = \frac{i\pi}{M}\ , \quad i = 0, 1, 2, \ldots\ , \tag{C17}$$

and from equation (C14):

$$\lambda = 2/3 + (1/3)\cos\frac{i\pi}{M}\ , \quad i = 0, 1, 2, \ldots\ . \tag{C18}$$

Then, substituting equations (C17) and (C18) into equation (C15), and noting that the sum of the infinity of solutions, as $i = 0, 1, 2, \ldots$, is also a solution, the complementary solution is obtained.

$$e_{m,n} = \sum_{i=0}^{\infty} C_i \cos\frac{i\pi m}{M}\left(\frac{2}{3} + \frac{1}{3}\cos\frac{i\pi}{M}\right)^{n}\ . \tag{C19}$$

In equation (C19), the constants C_i are to be evaluated from the arbitrary initial condition by making use of the orthogonality property of the cosine function. In view of the fact that in each term of the series in equation (C19), C_i is multiplied by a number not greater than unity, it is concluded that there is no build up of errors, and hence that equations (97) and (98) are stable. In fact, since

$$\cos\frac{i\pi m}{M}\left(\frac{2}{3} + \frac{1}{3}\cos\frac{i\pi}{M}\right)^n$$

will only occasionally equal unity, errors committed at a particular stage of the solution are expected to attenuate as the solution proceeds.

APPENDIX D

TRUNCATION ERROR FOR A SINGLE TIME STEP

In this section the truncation error committed in a single time step is estimated for the numerical algorithm of equations (97)-(102). This truncation error occurs as a result of the omission of P_1 and P_2' appearing in equations (93), (96), and (A13).

The quantity E_n is defined to be equal to the greatest absolute value of $e_{m,n}$ in the range $0 \leq m \leq M$, so that:

$$\left| e_{m,n} \right| \leq E_n \,, \tag{D1}$$

with the equality holding at at least one value of m. Then, using equations (C4), (103), and (D1), gives:

$$E_{n+1} \leq \frac{1}{6} E_n + \frac{4}{6} E_n + \frac{1}{6} E_n + Ph^6 \,,$$

or:

$$E_{n+1} - E_n \leq Ph^6 \,. \tag{D2}$$

Equation (D2) holds only in the range $m = -1, 0, 1, 2, \ldots, M-1$, since this is the range to which equation (C4) applies. (This range was extended in writing equations (C5)-(C7), by assuming that the surface temperature at $\xi = 1$ ($m = M$) is determined exactly through the numerical procedure. Such an assumption, though suitable for the purposes of Appendix C, is not applicable to the present section.)

For the point m = M on the boundary $\xi = 1$, the equations analogous to equations (C1) and (C2) are obtained from equations (99) and (A13):

$$t_{M,n+1} = D_1 + D_2\, t^2_{M,n+1}\,, \quad n = 0, 1, 2, \ldots\,, \tag{D3}$$

$$\bar{t}_{M,n+1} = \bar{D}_1 + \bar{D}_2\, \bar{t}_{M,n+1} + P_2'\,, \tag{D4}$$

$$n = 0, 1, 2, \ldots\,.$$

In equations (D3) and (D4), the quantities D_1 and D_2 are given by equations (100)-(102), while $\bar{D}_1$ and $\bar{D}_2$ are obtained from these same equations, using $\bar{t}_{M,n}$, $\bar{t}_{M,n-1}$, etc. As rough approximations, which however become close ones for sufficiently small values of h:

$$D_o \approx 3\,, \tag{D5}$$

$$D_1 \approx 1/3\,(t_{M-1,n} + 2t_{M,n})\,, \tag{D6}$$

$$\bar{D}_1 \approx 1/3\,(\bar{t}_{M-1,n} + 2\bar{t}_{M,n})\,, \tag{D7}$$

$$D_2 \approx \bar{D}_2 \approx 0\,. \tag{D8}$$

Then equations (D3) and (D4) become approximately:

$$t_{M,n+1} = (1/3)(t_{M-1,n} + 2t_{M,n}) , \tag{D9}$$

$$\bar{t}_{M,n+1} = (1/3)(\bar{t}_{M-1,n} + 2\bar{t}_{M,n}) + P_2' . \tag{D10}$$

The equation for the error, corresponding to equation (C4), now becomes:

$$e_{M,n+1} = \frac{e_{M-1,n} + 2e_{M,n}}{3} + P_2' . \tag{D11}$$

Using equation (D11) and equations (D1) and (103), equation (D2) is obtained. Thus, based on the very rough approximations embodied in equations (D5)-(D8), the same equation for error propagation holds for the boundary as for interior points.

With some reservations with regard to its validity at the boundary $\xi = 1$, the following upper bound for the truncation error committed in a single time step has been shown to hold:

$$E_{n+1} - E_n \leq Ph^6 . \tag{104}$$

APPENDIX E

SPECIFICATIONS OF EXPERIMENTAL EQUIPMENT

1. Vacuum Pump--Cenco Hyvac 14, Cat. No. 91705, Ser. No. 1120, speed 140 liters/min. at 1 atm, 1.05 liters/sec. at 1 micron mercury, ultimate pressure 0.1 micron mercury.
2. Multipoint Recording Potentiometer--Brown Electronik, Model No. SY153X89-(c)-II-II-16, Ser. No. P0155447001, range 0 to 5 and 0 to 55 mv $\pm$ 50 mv suppression, chart speed 24 in./hr.
3. McLeod Gage--Cenco, Cat. No. 94156, Ser. No. (glassware) 1423, dual scale 0 to 1000 micron mercury and 0 to 25 micron mercury.
4. Vacuum Chamber--length 24 in., inside diameter 8 in., constructed from std. 8 in. steel pipe with cast iron and cap, companion flange, and blind flange. Blind flange sealed to companion flange by "O" ring type seal. Thermocouple leads and heater leads taken through blind flange by means of Kovar ceramic seals mfd. by the Carborundum Co.
5. Heater--18 strands 26 gage chromel-"A" wire, resistance 2.5 ohm/ft., mfd. by Ogden Mfg. Co.
6. Thermocouples--Chromel-alumel, 28 gage, mfg. by Leeds

and Northrup Co.

7. Variable Transformers--Powerstat, type 116, mfd. by Superior Electric Co.

8. Sample--dimensions 6 in. by 16 in. by 3/4 in., made from Johns-Manville No. 352 asbestos shorts, molded to shape, with three thermocouples imbedded within it--one near the center and one imbedded slightly inside of each surface.

APPENDIX F

CRITERION FOR ESTIMATING EFFECT OF AMBIENT PRESSURE

Let $\tilde{t}(\xi,\tau)$ be the nondimensional temperature in the plate when conduction and convection effects are included in addition to the radiation boundary condition. Then the temperature $\tilde{t}$ must satisfy the system:

$$\frac{\partial \tilde{t}}{\partial \tau} = \frac{\partial^2 \tilde{t}}{\partial \xi^2}, \tag{F1}$$

$$\tilde{t} = \tilde{t}(\xi,\tau),\ 0 < \xi < 1,\ \tau > 0\ ;$$

$$\tilde{t}(\xi,0) = 1,\ 0 \leq \xi \leq 1\ ; \tag{F2}$$

$$\frac{\partial \tilde{t}}{\partial \xi}(0,\tau) = 0,\ \ \tau \geq 0\ ; \tag{F3}$$

$$\frac{\partial \tilde{t}}{\partial \xi}(1,\tau) = G - H\,\tilde{t}^4(1,\tau)$$

$$+ J - K\,\tilde{t}(1,\tau)\ ,\ \tau > 0\ . \tag{F4}$$

Let:

$$\tilde{t}(\xi,\tau) = t(\xi,\tau) + \hat{t}(\xi,\tau)\ , \tag{F5}$$

where $\hat{t}(\xi,\tau) << t(\xi,\tau)$, inasmuch as conduction and convection effects are assumed to be small in comparison with radiation effects. The temperature $t(\xi, \gamma)$ is that resulting from radiation effects only, and must satisfy equations (17)-(20). Substituting equation (F5) into equations (F1)-(F4) and utilizing equations (17)-(20), results in:

$$\frac{\partial \hat{t}}{\partial \gamma} = \frac{\partial^2 \hat{t}}{\partial \xi^2}, \tag{F6}$$

$$\hat{t} = \hat{t}(\xi, \gamma), \quad 0 < \xi < 1, \quad \gamma > 0;$$

$$\hat{t}(\xi, 0) = 0, \quad 0 \leq \xi \leq 1; \tag{F7}$$

$$\frac{\partial \hat{t}}{\partial \xi}(0, \gamma) = 0, \quad \gamma \geq 0; \tag{F8}$$

$$\frac{\partial \hat{t}}{\partial \xi}(1, \gamma) = J - K\, t(1, \gamma)$$

$$- 4Ht^3(1, \gamma)\, \hat{t}(1, \gamma), \quad \gamma > 0. \tag{F9}$$

In deriving equation (F9), $\hat{t}(1, \gamma)$ was neglected in comparison with $t(1, \gamma)$.

The temperature correction $\hat{t}$ due to the effect of conduction and convection must then satisfy equations (F6)-(F9). This system of equations can be readily solved, if $t(1, \gamma)$ in equation (F9) is replaced by a constant, say P_3. In this case, equation (F9) becomes:

$$\frac{\partial \hat{t}}{\partial \xi}(1,\gamma) = J - KP_3 - 4HP_3^{\,3}\,\hat{t}(1,\gamma)\ ,\quad \gamma > 0\ . \qquad \text{(F10)}$$

The system of equations (F6)-(F8), (F10) then constitutes a well-known convection problem whose solution is presented by Jakob (45).

The effect of replacing $t(1,\gamma)$ in equation (F9) by P_3 is shown qualitatively in Fig. 23. Curve I represents the solution to equations (F6)-(F9), where $\hat{t}$, due to the variation of $t(1,\gamma)$ with time, reaches a peak and then drops off, approaching the time axis asymptotically at large time. Curve II, on the other hand, shows the solution to equations (F6)-(F8), (F10), where $\hat{t}$ approaches asymptotically the value $(J - KP_3)/4HP_3^{\,3}$.

Inasmuch as curve I in Fig. 23 cannot be readily obtained quantitatively, curve II will be used as a criterion for assessing the effect of conduction and convection in the surrounding medium. The value which curve II approaches asymptotically will be used for this purpose. Thus, conduction and convection effects will be small if:

$$\frac{J - KP_3}{4HP_3^{\,3}} \ll 1\ . \qquad (132)$$

This equation is useful in deciding whether conduction and

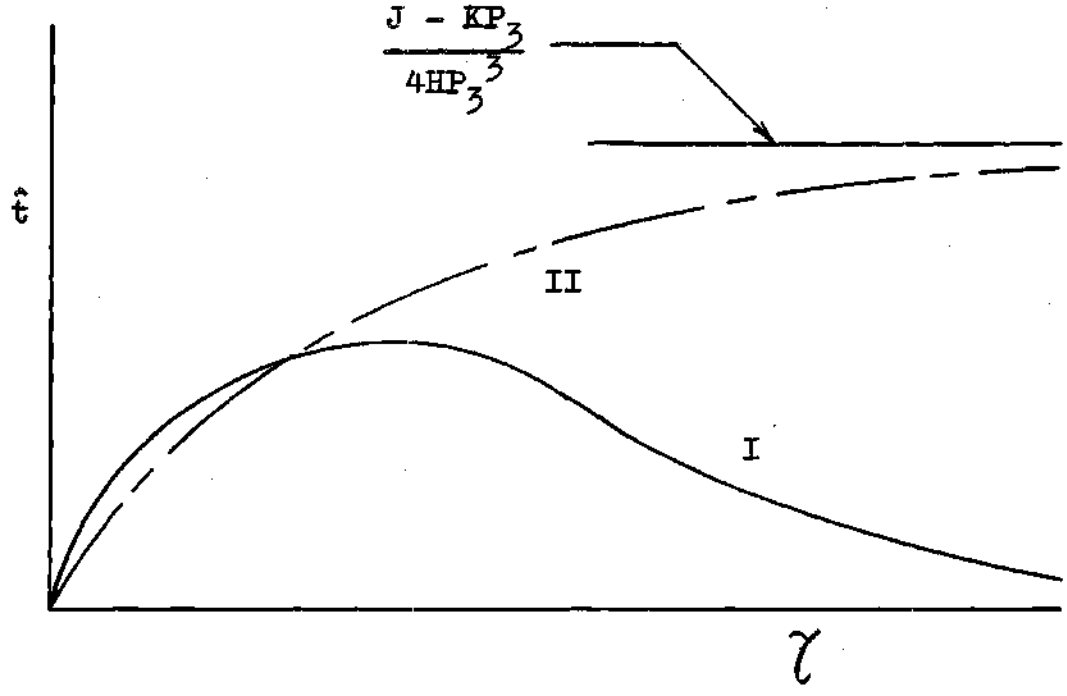

Figure 23

Prediction of Error due to Conduction Effects convection effects are negligible, but for obvious reasons cannot be used to correct for such effects.

Selecting the proper value of P_3 for use in equation (132) is difficult. The ideal value of P_3 is that value for which $(J - KP_3)/4HP_3^{\,3}$ equals the maximum value of curve I in Fig. 23. But this value can be determined only by computing curve I, which cannot readily be accomplished. A conservative choice for P_3 is that value between unity and t_e which causes the quantity:

$$U = \frac{J - KP_3}{4HP_3^{\,3}} \tag{F11}$$

to take on its greatest absolute value. The derivative of U is:

$$\frac{dU}{dP_3} = - \frac{3J - 2KP_3}{4HP_3^{\,4}} \, . \tag{F12}$$

For heating, $U > 0$, and also $1 \leq P_3 \leq J/K$. From equation (F12), it is seen that in this range $dU/dP_3 < 0$, and hence the maximum value of U occurs when $P_3 = 1$. For cooling, $U < 0$, and also $1 \geq P_3 \geq J/K$. Setting $dU/dP_3 = 0$, U has a minimum value when $P_3 = 3J/2K = 3t_2/2$. If $1 \geq 3t_e/2 \geq t_e$, then the minimum value of U (maximum absolute value of U) occurs when $P_3 = 3t_e/2$. Otherwise the minimum value of U occurs when $P_3 = 1$. Probably a more realistic choice for P_3 would be simply $P_3 = (1 + t_e)/2$.

APPENDIX G

NET EFFECT OF AMBIENT PRESSURE DURING EXPERIMENTAL ANALYSIS

In this section, the net effect of the presence of an ambient medium upon the value of $t(\xi, \gamma)$ measured during the experimental run, when the same ambient medium at the same pressure is present during the test run for measuring $\sigma \mathcal{F} L/k$, is considered. Denote the true temperature of the plate when conduction and convection in the ambient medium are considered by $\tilde{t}(\xi, \gamma)$, the temperature due to radiation alone by $t(\xi, \gamma)$ as previously, and let the deviation due to conduction and convection be denoted by $\hat{t}$. Then:

$$\tilde{t} = t + \hat{t} \, . \qquad \text{(G1)}$$

It is assumed that conduction and convection effects are sufficiently small that $\hat{t} \ll t$. The true temperature $\tilde{t}(\xi, \gamma)$ must satisfy the following system:

$$\frac{\partial \tilde{t}}{\partial \gamma} = \frac{\partial^2 \tilde{t}}{\partial \xi^2} \, , \quad 0 < \xi < 1 \, , \quad \gamma > 0 \, ; \qquad \text{(G2)}$$

$$\tilde{t}(\xi, 0) = 1 \, , \quad 0 \le \xi \le 1 \, ; \qquad \text{(G3)}$$

$$\frac{\partial \tilde{t}}{\partial \xi}(0,\gamma) = 0 \ , \quad \gamma \geq 0 \ ; \tag{G4}$$

$$\frac{\partial \tilde{t}}{\partial \xi}(1,\gamma) = \tilde{G} - \tilde{H}\,\tilde{t}^4(1,\gamma)$$

$$+ J - K\,\tilde{t}(1,\gamma) \ , \quad \gamma > 0 \ . \tag{G5}$$

In equation (G5), $\tilde{G}$ and $\tilde{H}$ are the true values:

$$\tilde{G} = G + \hat{G} \ , \tag{G6}$$

$$\tilde{H} = H + \hat{H} \ , \tag{G7}$$

while G and H are the values which hold under condition of radiation only at the boundary $\xi = 1$. It is assumed that $\hat{G} << \hat{G}$ and $\hat{H} << H$ in equations (G6) and (G7).

From the test for measuring $\sigma\mathcal{F} L/k$:

$$\frac{\sigma\mathcal{F} L}{k} = \frac{T_h - T_c}{2\sigma(T_c{}^4 - T_e{}^4)} \ . \tag{108}$$

When this is extended to allow for conduction and convection:

$$\frac{\widetilde{\sigma\mathcal{F}} L}{k} = \frac{T_h - T_c}{2\sigma(T_c{}^4 - T_e{}^4)} - \frac{CL(T_c - T_e)}{k\sigma(T_c{}^4 - T_e{}^4)} \ , \tag{G8}$$

where $\widetilde{\sigma\mathcal{F}}$ is the true value of the shape factor. Equations (108) and (G8) may be rewritten:

$$\frac{\sigma \mathcal{F} L}{k} = \frac{t_h - t_c}{2\sigma T_o^3(t_c^4 - t_e^4)} , \qquad (G9)$$

$$\frac{\tilde{\sigma} \mathcal{F} L}{k} = \frac{t_h - t_c}{2\sigma T_o^3(t_c^4 - t_e^4)} - \frac{K(t_c - t_e)}{\sigma T_o^3(t_c^4 - t_e^4)} . \qquad (G10)$$

Since $\sigma \mathcal{F} L/k$ and $\tilde{\sigma} \mathcal{F} L/k$ are theoretically independent of temperature, any value of t_c can be used in equations (G9) and (G10), t_h then automatically assuming the value necessary to satisfy equations (G9) and (G10). In light of this, replace t_c in equations (G9) and (G10) by $t(1,\gamma)$. This requires that t_h become a function of time, $t_h(\gamma)$. There results:

$$\frac{\sigma \mathcal{F} L}{k} = \frac{t_h(\gamma) - t(1,\gamma)}{2\sigma T_o^3\left[t^4(1,\gamma) - t_e^4\right]} , \qquad (G11)$$

$$\frac{\tilde{\sigma} \mathcal{F} L}{k} = \frac{t_h(\gamma) - t(1,\gamma)}{2\sigma T_o^3\left[t^4(1,\gamma) - t_e^4\right]} - \frac{K\left[t(1,\gamma) - t_e\right]}{\sigma T_o^3\left[t^4(1,\gamma) - t_e^4\right]} . \qquad (G12)$$

Using the definitions of G and H, equations (21) and (22) respectively, and of $\tilde{G}$ and $\tilde{H}$, and making use of equations (G6), (G7), (G11), and (G12), one finally arrives at the result:

$$\hat{G} = -K \frac{t(1,\gamma) - t_e}{t^4(1,\gamma) - t_e^4} t_e^4 , \tag{G13}$$

$$\hat{H} = -K \frac{t(1,\gamma) - t_e}{t^4(1,\gamma) - t_e^4} . \tag{G14}$$

Utilizing equations (G1), (G6), (G7), (G13), and (G14) in equation (G5), gives:

$$\frac{\partial t}{\partial \xi}(1,\gamma) + \frac{\partial \hat{t}}{\partial \xi}(1,\gamma) = G - H\left[t(1,\gamma) + \hat{t}(1,\gamma)\right]^4 - K\frac{t(1,\gamma)-t_e}{t^4(1,\gamma)-t_e^4}\left\{t_e^4 - \left[t(1,\gamma) + \hat{t}(1,\gamma)\right]^4\right\} + J - K\left[t(1,\gamma) + \hat{t}(1,\gamma)\right] . \tag{G15}$$

Making use of equation (20), and taking account of the fact that $\hat{t} << t$, equation (G15) reduces to:

$$\frac{\partial \hat{t}}{\partial \xi}(1,\gamma) = -4H\, t^3(1,\gamma)\, \hat{t}(1,\gamma) . \tag{G16}$$

In obtaining equation (G16), use is also made of the fact that $J = Kt_e$. Combining equations (17)-(19), (G1), and (G2)-(G4) gives:

$$\frac{\partial \hat{t}}{\partial \gamma} = \frac{\partial^2 \hat{t}}{\partial \xi^2} , \quad 0 < \xi < 1, \quad \gamma > 0 ; \tag{G17}$$

$$\hat{t}(\xi, 0) = 0 \ , \quad 0 \leq \xi \leq 1 \ ; \tag{G18}$$

$$\frac{\partial \hat{t}}{\partial \xi}(0, \gamma) = 0 \ , \quad \gamma \geq 0 \ . \tag{G19}$$

The temperature deviation caused by conduction and convection effects in the ambient medium must then satisfy equations (G16)-(G19). The solution to this system of equations is:

$$\hat{t}(\xi, \gamma) \equiv 0 \ . \tag{G20}$$

Thus, subject to the assumptions used in deriving equations (G16)-(G19), it is concluded that the presence of an ambient medium has no effect upon the temperature history $t(\xi, \gamma)$ as measured in the experimental analysis. These assumptions are valid only for sufficiently small ambient pressure. No quantitative estimate of the variation of error introduced by these assumptions with pressure has been made in this work.

APPENDIX H

DATA FROM SHAPE FACTOR AND DIFFUSIVITY TESTS

Shape Factor.--The data taken during the tests for measuring the shape factor over the temperature range 80-500°F. are tabulated in Table 3. The shape factor combination $\sigma \mathcal{F} L/k$ was computed from the equation:

$$\frac{\sigma \mathcal{F} L}{k} = \frac{T_h - T_c}{2\sigma (T_c{}^4 - T_e{}'^4)} \quad . \tag{109}$$

Table 3. Data from Shape Factor Test

No.	Pressure μ Hg.	T_h °F	T_c °F	T_e °F	T_e' °F	$\frac{\sigma \mathcal{F} L}{k}$
red side						
1	1.7	148.0	120.3	76.7	76.7	0.260
2	1.7	239.5	170.3	76.7	76.7	0.269
3	1.9	332.0	217.3	76.3	76.3	0.261
4	2.7	408.5	253.0	76.3	76.3	0.258
5	4.1	471.0	431.0	77.5	411.5	0.209
6	2.6	455.7	369.0	76.7	321.0	0.258
7	2.6	437.0	317.0	77.0	225.0	0.241
8	3.7	454.0	412.0	78.0	390.5	0.223
9	3.3	480.5	447.0	76.3	431.3	0.217

(Continued)

Table 3. Data from Shape Factor Test (Continued)

No.	Pressure μ Hg.	T_h °F	T_c °F	T_e °F	T_e' °F	$\frac{\mathcal{F} L}{k}$
<u>black side</u>						
1	1.7	152.5	122.0	77.0	77.0	0.278
2	1.8	153.0	123.0	76.3	76.3	0.269
3	1.7	247.5	177.5	75.0	75.0	0.246
4	1.8	247.5	178.0	76.7	76.7	0.246
5	2.1	349.0	228.0	76.0	76.0	0.250
6	2.1	431.0	268.5	76.3	76.3	0.240
7	3.3	472.5	431.0	76.3	410.0	0.212
8	2.5	457.0	374.0	77.5	326.0	0.269
9	2.7	445.0	327.5	79.0	235.5	0.228

<u>Diffusivity</u>.--In Table 4 are tabulated the data taken during the test runs for measuring diffusivity. The values for T_1, T_2, T_3, T_4, and θ_α were taken from the chart printed by the recording potentiometer. The values of α/L^2 were computed from the given data by solving equations (122)-(124), for which purpose a computer program was written.

Table 4. Data from Diffusivity Test

No.	Pressure μ Hg.	T_1 °F	T_2 °F	T_3 °F	T_4 °F	θ_α hr.	$\frac{T_1+T_3}{2}$ °F	α/L^2 hr.$^{-1}$
1	1.6	80.0	127.0	96.8	95.0	0.202	87.5	6.86
2	1.8	127.0	176.7	141.5	137.0	0.1168	132.0	6.44
3	1.9	176.7	234.0	202.8	195.0	0.173	185.8	5.76
4	2.0	234.0	319.0	271.5	255.0	0.120	244.5	5.47
5	2.2	318.5	396.5	354.8	339.0	0.1116	328.8	5.33
6	3.2	398.5	465.5	434.6	419.0	0.1162	408.8	5.25
7	4.1	464.5	519.0	495.5	484.0	0.1246	474.2	5.29

Sample calculations for run No. 1 are presented, illustrating the method of computing the correction for deviation from step-function boundary condition:

Run No. <u>1</u> Date <u>7-20-62</u>

Pressure <u>1.6</u> μHg.

A = <u>0.667</u> ft.^{2} A_w = <u>0.0871</u> ft.^{2}

$c_w \rho_w V_w$ = <u>0.00165</u> Btu/°R ϵ_w = <u>0.65</u>

$\mathcal{F}'$ = <u>0.90</u>

$K_1 = 2A\mathcal{F}'/\epsilon_w A_w$ = <u>21.2</u>

R = (red) <u>11.8</u> (black) <u>12.3</u> ohm

θ_1 = <u>10</u> sec. <u>0.00278</u> hr.

$\sigma A \mathcal{F}'/1.706$ = <u>6×10^{-10}</u>

$(c_w \rho_w V_w)R/3.413\ \theta_1$ = (red) <u>2.05</u> (black) <u>2.14</u>

T_1 = <u>1.06</u> mv <u>80.0</u> °F <u>540.0</u> °R

T_e = 1.06 mv 80.0 °F 540.0 °R

v_1 = (red) 0 (black) 0 volts

Expected T_2 = 140 °F 600 °R

Expected $P_{w2} = P_{w1} + (\sigma A \mathcal{F}'/1.706)(T_2^4 - T_1^4)$

= (red) 27 (black) 27 watts

Expected $v_2 = \sqrt{P_{w2}R}$ = (red) 17.8 (black) 18.2 volts

$T_{w1} = \left[K_1(T_1^4 - T_e^4) \right]^{1/4}$ = 80 °F 540 °R

Expected $T_{w2} = \left[K_1(T_2^4 - T_e^4) \right]^{1/4}$ = 362 °F 822 °R

$v' = \left[v_2^2 + (c_w \rho_w V_w R/3.413\ \theta_1)(T_{w2} - T_{w1}) \right]^{1/2}$

= (red) 30 (black) 30.6 volts

APPENDIX I

DATA FROM EXPERIMENTAL RUNS

The sample calculations necessary for computing the correction $\bar{\theta}$ to allow for the effect of deviation from step-function boundary condition, as well as the computation of the parameters G and H, are as follows:

Run No. 3 Date 7-22-62

Pressure (initial) 2.8 (final) 2.6 μHg.

$c_w \rho_w V_w$ = 0.00165 Btu/°R

ϵ_w = 0.65

A_w = 0.0871 ft.^{2}

$\mathcal{F}'$ = 0.90

$K_1 = 2A\mathcal{F}/\epsilon_w A_w$ = 21.2

$K_2 = c_w \rho_w V_w / \sigma \epsilon_w A_w$ = 1.7×10^7 hr. °R^3

V_o = (red) 64.6 (black) 63.8 volts

T_o = 7.74 mv 374.5 °F 834.5 °R

Initial paper scale reading = 9

T_e = (initial) 1.06 mv 80.0 °F 540.0 °F (final) 1.07 mv 80.3 °F 540.3 °F (average) 80.1 °F 540.1 °R

$T_{av} = (T_o + T_{eave})/2$ = 227.3 °F 687.3 °R

α/L^2 at T_{av} = 5.78 hr.$^{-1}$

$\mathcal{F}L/k$ at T_{av} = $\underline{0.255}$

$G = \sigma(\mathcal{F}L/k)\ T_{eave}{}^4/T_o$ = $\underline{0.044}$

$H = \sigma(\mathcal{F}L/k)\ T_o{}^3$ = $\underline{0.254}$

$T_{wo} = \left[K_1(T_o{}^4 - T_e{}^4)\right]^{1/4}$ = $\underline{1240}$°F $\underline{1700}$ °R

$\bar{\theta} = K_2(T_{wo} - T_e)/(T_{wo}{}^4 - T_e{}^4)$ = $\underline{0.00235}$ hr.

1 unit of paper scale = 0.0833 hr.

The values of temperature as a function of time in the experimental run were read from the recording potentiometer chart. Some sample values taken from this chart for Run No. 3, together with the calculations necessary to reduce the data to the values plotted in Fig. 16, are tabulated in Table 5. In this table, three points are selected at random at each of the two faces of the sample, and at the center, and the necessary calculations indicated. In these calculations, the temperature in millivolts is converted to °F by means of the table provided for a chromel-alumel thermocouple. The values of γ and t are obtained from the equations:

$$\gamma = \frac{\alpha(\theta - \bar{\theta})}{L^2},$$

$$t = \frac{T}{T_o}.$$

Table 5. Data from Experimental Run No. 3

No.	Paper Scale	θ hr.	$\theta - \bar{\theta}$ hr.	Temperature mv	°F	°R	γ	t
red side								
1	0.06	0.0050	0.0026	7.54	365.5	825.5	0.015	0.990
7	0.46	0.0384	0.360	6.63	324.3	784.3	0.208	0.939
17	7.06	0.589	0.587	2.82	157.0	617.0	3.39	0.739
black side								
4	0.23	0.0192	0.0168	7.04	343.0	803.0	0.097	0.963
9	0.63	0.0525	0.0501	6.43	315.3	775.3	0.290	0.930
22	14.02	1.170	1.168	1.70	108.0	568.0	6.75	0.681
center								
2	0.21	0.0175	0.0151	7.72	373.5	833.5	0.087	0.999
8	0.61	0.0509	0.0485	7.46	362.0	822.0	0.280	0.986
16	1.87	0.156	0.154	6.14	302.3	762.3	0.890	0.915

APPENDIX J

ESTIMATE OF INNER WALL SURFACE TEMPERATURE

An estimate of the temperature of the inner surface of the vacuum chamber wall is made in this section, based on equations (134)-(137). Although not shown in Fig. 7, the water jacket was baffled, the cross section of the flow passage between the baffle plates being approximately 1.5 in. by 6 in. The computations made in estimating the temperature of the inner wall surface $\widetilde{T}_e$, and the estimate of the error made in replacing its value by the cooling water temperature T_e, are summarized in Table 6. The value of the coefficient C_c in equation (137) was obtained from Kreith (47). Computations are shown for runs 1, 2, and 3; those for runs 3', 3a, 3b, and 3c are omitted since they are almost identical with Run No. 3.

Table 6. Estimate of Inner Wall Temperature

Run No.	v volts	T_e °F	ΔT_{H_2O} °	$\dot{Q}$ Btu/hr.	$\dot{m}$ lb./hr.	T_e °F	T_o °F	$\frac{T_e^4 - T_e^4}{T_o^4}$
1	90	81.0	0.5	2300	≈ 5000	90.5	504.5	0.007
2	30	79.4	≈ 0	255	≈ 5000	80.4	186.7	0.006
3	64	80.1	0.2	1160	≈ 5000	84.9	374.5	0.006

Dimensions of flow passage--1.5" by 6"; hydraulic diameter 0.2'.

Velocity of water = 0.36 ft./sec.

Reynolds number = 7700

C_c = 61.1 Btu/hr.ft.^{2}F

A_c = 4.18 ft.^{2}, δ = 0.021' , k_c = 26 Btu/hr.ft.F (48)

APPENDIX K

COMPUTATION OF J AND K FOR EXPERIMENTAL RUNS 3b AND 3c

Run No. 3c.--In this analysis the free convection effects in the ambient medium are treated by first replacing the sample and vacuum chamber wall system by two parallel plates 6" wide by 16" tall and 3" apart, side effects being neglected. The distance separating the plates was arbitrarily chosen to be 3", since it is extremely difficult to determine what the proper distance should be, due to the complexity of the free convection effects.

In estimating the heat transfer coefficient, the initial sample temperature was used for the high temperature plate, since this gives the largest heat transfer coefficient and hence is conservative. The heat transfer coefficient was obtained from Kreith (49). A summary of the calculations is given in Table 7.

Run No. 3b.--In this run, at a pressure of 250 μ Hg., free convection effects can be shown to be negligible. Also from equations (130) and (131), the value of C for molecular conduction (curve I of Fig. 9) far exceeds the value of C based on atmospheric pressure, so that C is located on curve II of Fig. 9. As in the previous case, the system of sample and vacuum chamber wall is replaced by two parallel

Table 7. Computation of J and K for Run No. 3c

$T_o = 374.0$ F ; $T_e = 75$ F ; $\Delta T = 299$ F

$k_m = 0.0179$ Btu/hr.ft.F ; $N_{pr} = 0.72$

$b = 3" = 0.25'$; $L = 16" = 1.33'$

$N_{Gr} = 3.5\text{x}10^6$; $N_{Gr}\, N_{pr}\, \frac{b}{L} = 0.472\text{x}10^6$

$N_{Nu} = 15.0$ (from Kreith (49))

$C = 1.07$ Btu/hr.ft.^{2}F

$$K = \frac{CL}{k} = 0.338$$

$$J = K\,\frac{T_e}{T_o} = 0.218$$

For $P_3 = 3t_e/2 = 0.97$:

$$\frac{J - KP_3}{4HP_3^{\,3}} = -0.118$$

For $P_3 = \dfrac{1 + t_e}{2} = 0.824$:

$$\frac{J - KP_3}{4HP_3^{\,3}} = -0.107$$

plates 6" by 16", side effects being neglected. The plates are spaced a distance b apart. From a flux plot (not shown) drawn for the real situation, the value of b was found to be approximately 1.5". The calculations are summarized in Table 8.

Table 8. Computation of J and K for Run No. 3b

k_m = 0.0179 Btu/hr.ft.F (from Kreith (50))

$b = 1.5" = 0.125'$

$C = \frac{km}{b} = 0.124$ Btu/hr.ft.^{2}F

$K = \frac{CL}{k} = 0.045$

$J = Kt_e = 0.0281$

For $P_3 = 3t_e/2 = 0.97$:

$$\frac{J - KP_3}{4HP_3^3} = -0.0167$$

For $P_3 = \frac{1 + t_e}{2} = 0.824$:

$$\frac{J - KP_3}{4HP_3^3} = -0.0158$$

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In 1950 he entered the University of Tennessee and received the degree Bachelor of Science in Mechanical Engineering in 1954. He immediately entered the Graduate School of the University of Tennessee as a graduate assistant, to work toward the degree Master of Science in the field of mechanical engineering, which degree was awarded in 1958.

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